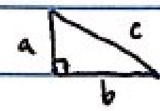
## Week 1 Lecture – College Algebra

The Pythagorean Theorem : $a^2+b^2=c^2 \Leftrightarrow$ 



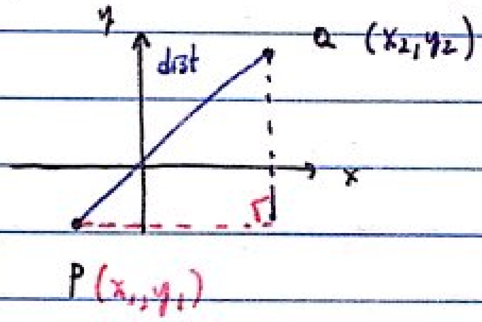
Counterexample: $1^2+2^2 \neq 3^2$

Example: Solve for the missing leg: 



$$a^2+5^2=13^2 \Rightarrow a^2=144 \Rightarrow a=\pm 12, \quad \boxed{a=12.}$$

Application 1. The distance formula





$$\Delta x^2 + \Delta y^2 = dist^2$$

$$(x_2-x_1)^2 + (y_2-y_1)^2 = dist^2$$

so, $\boxed{dist = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$ *

Ex. $P(-3,4)$, $Q(5,-2)$, find $dist(P,Q)$.

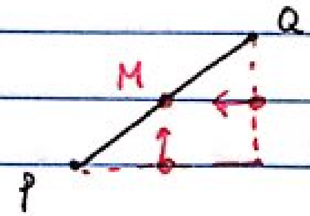
$$dist = \sqrt{(5-(-3))^2+(-2-4)^2}$$
$$= \sqrt{8^2+(-6)^2}$$
$$= \sqrt{64+36}$$
$$= \sqrt{100}$$
$$= 10$$

Ex. $P(3,6)$, $Q(-3,0)$, find $dist(P,Q)$.

$$dist = \sqrt{(-3-3)^2+(0-6)^2}$$
$$= \sqrt{(-6)^2+(-6)^2}$$
$$= \sqrt{72}$$
$$= \sqrt{2\cdot 36}$$
$$= \sqrt{2}\sqrt{36}$$
$$= 6\sqrt{2}$$

<u>Midpoint formula</u> $P(x_1,y_1)$, $Q(x_2,y_2)$, the <u>midpoint</u> is given by

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$



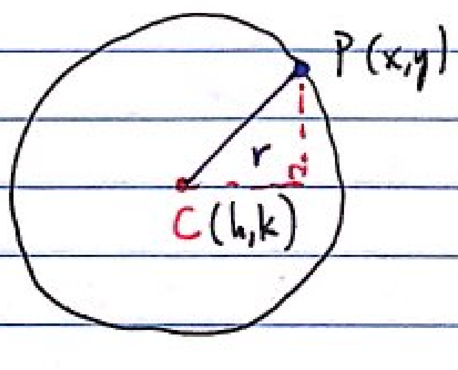

<u>Ex.</u> $P(3,6)$, $Q(-3,0)$, find the midpoint of the segment connecting the points.

$$M = \left(\frac{3-3}{2}, \frac{6+0}{2}\right) = \left(\frac{0}{2}, \frac{6}{2}\right) = (0,3)$$

<u>Application 2.</u> <u>Equation of a Circle</u>

<u>Def'n.</u> A <u>circle</u> is the set of all points in a plane equidistant from a fixed point in the plane.

The fixed point is the <u>center</u>.
The distance is called the <u>radius</u> of the circle.




Using the <u>Pythagorean Thm</u>:

$$\boxed{(x-h)^2 + (y-k)^2 = r^2} \quad *$$

$(h,k)$ = center
$r > 0$ radius

Ex. $(h,k) = (-3,4)$ $\quad r = 6$

$$(x+3)^2 + (y-4)^2 = 6^2$$
$$(x+3)^2 + (y-4)^2 = 36$$

Ex. Center = origin $\quad$ radius = 13.

$$(x-0)^2 + (y-0)^2 = 13^2$$
$$x^2 + y^2 = 169$$

Ex. $(h,k) = (3,4)$ and the origin lies on the circle.

$$r = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$(x-3)^2 + (y-4)^2 = 25$$

$\boxed{\text{FTIS}}$ Ex. The points $(0,0)$ and $(-6,8)$ are the endpoints of a diameter of the circle.

$$d = \sqrt{36+64} = \sqrt{100} = 10, \text{ so } r = 5$$
$$C = M = (-3,4), \text{ so }$$
$$(x+3)^2 + (y-4)^2 = 25$$

## Completing the Square

Consider an expression of the form $ax^2 + bx + c$.
We will frequently need to get this in the form
$$a(x-h)^2 + k.$$

to do this, $h = \frac{-b}{2a}$ and $k = c - \frac{b^2}{4a}$

The process goes as follows:

$$ax^2 + bx + c$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$
$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{1}{a}\left(c - \frac{b^2}{4a}\right)\right)$$
$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Example. $x^2 + 6x - 3 = (x^2 + 6x + 9) - 3 - 9$

$= (x+3)^2 - 12$ ✓

$a = 1$

$b = 6$

$\frac{b}{2} = 3$

$\left(\frac{b}{2}\right)^2 = 9$

Example. $2x^2 + 4x - 6 = 2(x^2 + 2x - 3)$

$= 2(x^2 + 2x + 1 - 3 - 1)$

$= 2(x+1)^2 + 2(-4)$

$= 2(x+1)^2 - 8$

$a = 2$

$b = 4$

$\frac{b}{2a} = \frac{4}{4} = 1$

$\left(\frac{b}{2a}\right)^2 = 1$

$c = -6$

$c - \frac{b^2}{4a} = c - a\left(\frac{b}{2a}\right)^2 = -6 - 2(1) = -6 - 2 = -8$

Another Example. $x^2 + 3x + 5$

$= \left(x^2 + 3x + \frac{9}{4}\right) + 5 - \frac{9}{4}$

$= (x + 3/2)^2 + 11/4$

$b = 3$

$\frac{b}{2} = \frac{3}{2}$

$\left(\frac{b}{2}\right)^2 = \frac{9}{4}$

Briefly back to circles:

An equation of the form $Ax^2 + By^2 + Cx + Dy = E$ is a circle if and only if it can be rewritten in the form

$A(x-h)^2 + A(y-k)^2 = r^2$

by completing squares on both $x$ and $y$.

In particular, $A=B$.

Example. $\boxed{x^2+6x} + \boxed{y^2-8y} = 24$

$x^2+6x$ | $y^2-8y$
$x^2+6x+9-9$ | $y^2-8y+16-16$
$(x+3)^2-9$ | $(y-4)^2-16$

So,

$$x^2+6x+y^2-8y = 24 \quad \text{becomes}$$
$$(x+3)^2 - 9 + (y-4)^2 - 16 = 24$$
$$(x+3)^2 + (y-4)^2 = 24+9+16$$
$$(x+3)^2 + (y-4)^2 = 49$$

So $C = (-3,4)$ and $r = 7$

More examples of CtS.

$$\begin{cases} x^2 + 20x + 100 \\ 3y^2 - 9y + 27 \\ 13z^2 + 169z \end{cases}$$