Section 3.1—Complex Numbers

The Imaginary Unit i

$i = \sqrt{-1}$

or $i^2 = -1$

The Expression $\sqrt{-a}$

if $a > 0$,

$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$

e.g.

$\sqrt{-9} = i\sqrt{9} = 3i$

$\sqrt{-25} = i\sqrt{25} = 5i$

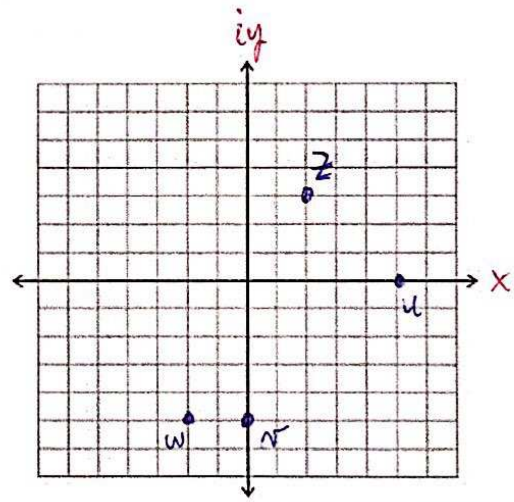
Graphical Interpretation

A complex number is of the form

$$\boxed{z = x + iy}$$

where x and y are real numbers.

To each complex number $x+iy$ we associate the point (x,y) in the xy-plane (now thought of as the complex plane).



x = "real part"
x-axis = $\mathbb{R}$
y = "imaginary part"
y-axis = pure imaginary #

Ex. $z = 2+3i \rightarrow (2,3)$ $\quad u = 5$

$w = -2-5i \rightarrow (-2,-5)$ $\quad v = -5i$

Complex Arithmetic—Addition and Subtraction

Let $z = a+bi$ and $w = c+di$

Then

$$z+w = (a+bi)+(c+di) = (a+c)+(b+d)i$$

$$z-w = (a+bi)-(c+di) = (a-c)+(b-d)i$$

"Combine like terms"

Multiplication

The product is

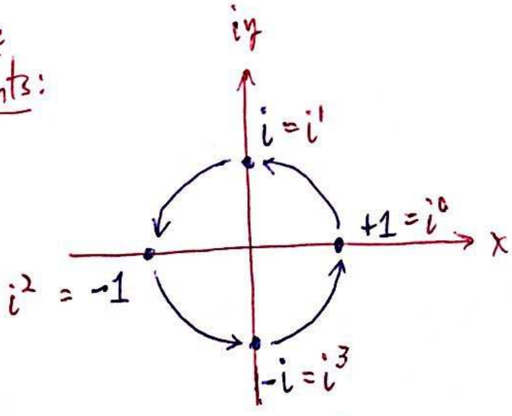
$$z \cdot w = (a+bi)(c+di) = ac + adi + bci + bd\overset{-1}{i^2}$$

$$= (ac-bd) + (ad+bc)i$$

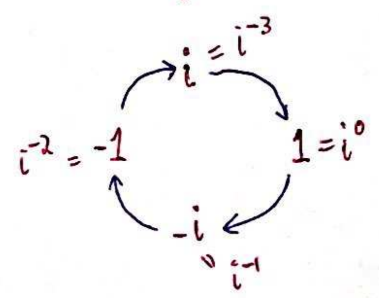
Idea: "FOIL"

The Circle of i's

Positive Exponents:



Negative Exponents:



Idea: Find the remainder of the exponent divided by 4, and then refer to the circle.

Ex. $i^{47} = i^{44+3} = i^3 = -i$

Remainder is 3.

$i^{-27} = i^{-24-3} = i^{-3} = i$

"Remainder" is -3 (or $+1$)

Example Compute

(*a*) $(5-4i)(7-2i)$

$= 35 - 10i - 28i + 8i^2$

$= 35 - 8 \quad -10i - 28i$

$= \boxed{27 - 38i}$

(*b*) $(6+5i)(6-5i)$

$= 36 - 25i^2$

$= 36 + 25$

$= \boxed{61}$

(*c*) $(4+3i)^2$

$= 16 + 24i + 9i^2$

$= 16 - 9 + 24i$

$= \boxed{7 + 24i}$

(*d*) i^{13}

$= i^{12+1} = i^1 = i$

(*e*) i^{56}

$= i^{56+0} = i^0 = 1$

(*f*) i^{-3}

$= i^{-3+4} = i^1 = i$

Complex Conjugates

If $z=a+bi$, then its <u>complex conjugate</u> is $\boxed{\bar{z}=a-bi}$.

$$z\cdot\bar{z}=(a+bi)(a-bi)$$
$$=a^2-(bi)^2$$
$$=a^2-b^2i^2$$
$$=a^2+b^2 \leftarrow \text{real, non-negative!}$$

Dividing Complex Numbers

$$\frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Example Compute

(a) $\dfrac{(7-i)}{(3+i)}\dfrac{(3-i)}{(3-i)}=\dfrac{21-7i-3i+i^2}{3^2+1^2}=\dfrac{20-10i}{10}=\dfrac{20}{10}-\dfrac{10}{10}i$

$=\boxed{2-i}$

(b) $\dfrac{(3+2i)}{(5-i)}\dfrac{(5+i)}{(5+i)}=\dfrac{15+3i+10i+2i^2}{5^2+1^2}=\dfrac{13+13i}{26}=\dfrac{13}{26}+\dfrac{13}{26}i$

$=\boxed{\frac{1}{2}+\frac{1}{2}i}$

(c) $\dfrac{3}{i}=3i^{-1}=\boxed{-3i}$