Section 3.3—Quadratic Equations and Inequalities

Quadratic Equation in one variable

A quadratic equation is of the form

$$ax^2 + bx + c = 0.$$

The Principle of Zero Products

If we can factor the quadratic equation as

$$a(x-\alpha)(x-\beta) = 0$$

then the solutions are $x = \alpha$ and $x = \beta$.

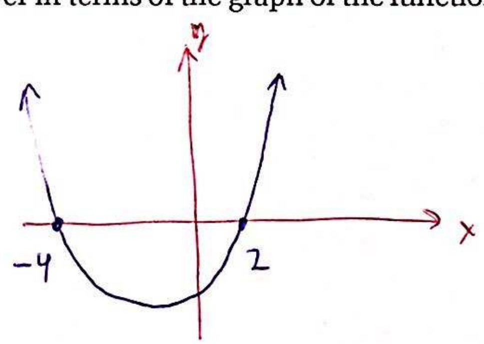
Example Solve $2x^2 + 4x - 16 = 0$. Interpret your answer in terms of the graph of the function.

$2(x^2 + 2x - 8) = 0$

$2(x+4)(x-2) = 0$

$\boxed{x = -4,\ x = 2}$

Two real solutions, distinct!



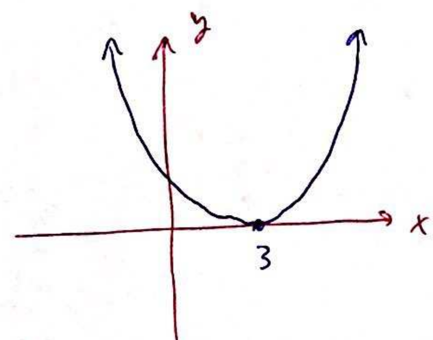
Example Solve $x^2 - 6x + 9 = 0$. Interpret your answer in terms of the graph of the function.

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$\boxed{x = 3}$

One real solution



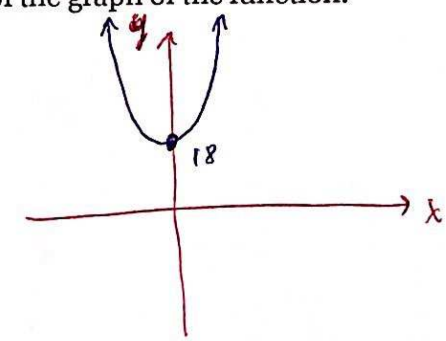
Example Solve $2x^2 + 18 = 0$. Interpret your answer in terms of the graph of the function.

$2(x^2+9) = 0$

$x^2 = -9$

$\boxed{x = \pm 3i}$

No real solutions.
Solutions are complex conjugates.



Summarize the possibilities

One real solution	Vertex of graph lies on x-axis.
Two distinct real solutions	Graph crosses x-axis in 2 points.
No real solutions (Two complex solutions)	Graph does not cross the x-axis.

Examples Solve by completing the square.

(*a*) $x^2 - 4x + 2 = 0$

$(x^2 - 4x + 4) + 2 - 4 = 0$

$(x-2)^2 = 2$

$x - 2 = \pm\sqrt{2}$

$\boxed{x = 2 \pm \sqrt{2}}$ Two real solutions.

(*b*) $\frac{8x^2}{8} + \frac{2x}{8} = \frac{3}{8}$

$x^2 + \frac{1}{4}x + \frac{1}{64} = \frac{3 \cdot 8}{8 \cdot 8} + \frac{1}{64}$

$(x + \frac{1}{8})^2 = \frac{25}{64}$

$x + \frac{1}{8} = \pm \frac{5}{8}$

$x = -\frac{1}{8} \pm \frac{5}{8}$

$\boxed{x = -\frac{3}{4}, \quad x = \frac{1}{2}}$

Thought Exercise Given a quadratic equation in standard form $ax^2 + bx + c = 0$, we can always solve by completing the square.

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right) = 0$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$\vdots$

Finish this for Extra Credit.

Continued... The end result is

The Quadratic Formula:

$$\boxed{x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}}$$

Example Solve $x^2 + 6x + 2 = 0$.

$a = 1, \; b = 6, \; c = 2$

$$x = \frac{-6}{2(1)} \pm \frac{\sqrt{6^2 - 4(1)(2)}}{2(1)} = -3 \pm \frac{\sqrt{36-8}}{2} = -3 \pm \frac{\sqrt{28}}{2} = -3 \pm \frac{\sqrt{7}\sqrt{4}}{2}$$

$$\boxed{x = -3 \pm \sqrt{7}}$$

The Discriminant Test

The discriminant is $b^2 - 4ac$ (the radicand of the Quad. F.).

Test:

$b^2 - 4ac > 0$	Two distinct real sol'ns
$b^2 - 4ac = 0$	One real solution
$b^2 - 4ac < 0$	No real solutions (2 complex)

Example Solve analytically, then sketch a graph.

$$x^2 = 5x - 4$$

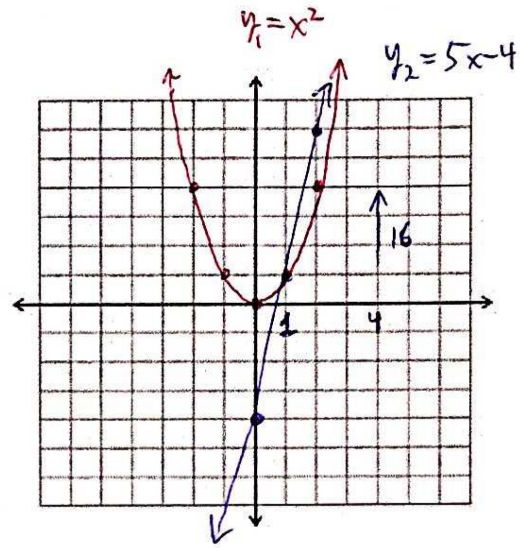
$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$\boxed{x=1,\ x=4}$

$y_1 = x^2$

$y_2 = 5x - 4$



Quadratic Inequalities

To solve $ax^2 + bx + c < 0$ $(>, \leq, \geq)$

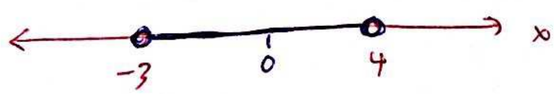
first solve $ax^2 + bx + c = 0$, then plot solutions on a number line and choose a test value. Shade accordingly and interpret your solution as an interval.

Example Solve $x^2 - x - 12 < 0$, and plot the solution on a number line.

$x^2 - x - 12 = 0$

$(x-4)(x+3) = 0$

$x = -3,\ x = 4$



$0^2 - 0 - 12 = -12 < 0$ ✓

So shade the middle.

$\boxed{\text{The solution is } (-3, -4).}$

Example Solve $2x^2 \geq -5x + 12$. Plot the solution on a number line.

$$2x^2 + 5x - 12 \geq 0$$

Solve: $2x^2 + 5x - 12 = 0$

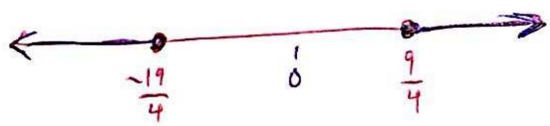
$$\cancel{2}\left(x^2 + \frac{5}{2}x + \frac{25}{4} - 6 - \frac{25}{4}\right) = 0$$

$$\left(x + \frac{5}{4}\right)^2 = \frac{49}{4}$$

$$x = -\frac{5}{4} \pm \frac{7}{2}$$

$$x = \frac{9}{4}, \frac{-19}{4}$$

$$\boxed{\text{The Solution is } \left(-\infty, -\frac{19}{4}\right] \cup \left[\frac{9}{4}, \infty\right)}$$



$$2(0)^2 \geq -5 \cdot 0 + 12$$

$0 \geq 12$ ✗ No. Shade outside

Example Solve $x^2 + x + 20 < 0$. Plot the solution on a number line.

$$x^2 + x + 20 = 0$$

$$x^2 + x + \frac{1}{4} = -20 + \frac{1}{4}$$

$$\left(x + \frac{1}{2}\right)^2 = -\frac{79}{4}$$

$$x = -\frac{1}{2} \pm \frac{\sqrt{79}\, i}{2}$$

No real solutions, so the inequality is either always true, or never true.

Check $x = 0$.

$$0^2 + 0 + 20 < 0$$

$20 < 0$ ✗ No.

So this inequality has $\boxed{\text{NO SOLUTION.}}$