MA 135: College Algebra (Fall 2016)
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Week 9 Notes

Butler Community College

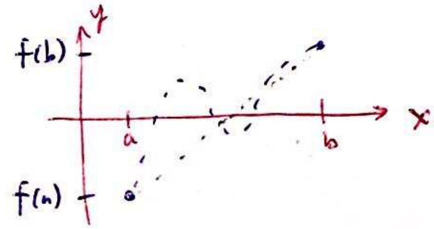
How to use this handout—This handout contains a skeleton of the notes that we will study in class this week. I've typed out definitions and theorems so that you don't have to exasperatedly copy what I'm writing, and populated these pages with a number of examples. My expectation of you is that you will fill in all of the details, ideas, *etc*, that I've left out.

Section 3.6—Theory of Polynomials, part I

1. Intermediate Value Theorem

Let f be a continuous function on the interval $[a,b]$. If $f(a)$ and $f(b)$ have opposite signs, then f has a real zero between a and b.

In particular, every polynomial P w/ real coefficients is continuous on its entire domain, so this theorem applies!



2. Example Show that the polynomial function $P(x) = x^3 - 2x^2 - x + 1$ has a real zero between 2 and 3.

$P(2) = 2^3 - 2(2)^2 - 2 + 1 = 8 - 8 - 2 + 1 = -1 < 0$

$P(3) = 3^3 - 2(3)^2 - 3 + 1 = 27 - 18 - 3 + 1 = 7 > 0$

Since $P(2) < 0 < P(3)$, then P must have a zero between 2 and 3 by the Intermediate Value Theorem (IVT)!

3. Division Algorithm for Polynomials

Let P and D be polynomial functions such that $0 < \deg(D) < \deg(P)$. Then there exist unique polynomials Q and R such that

$\boxed{P(x) = D(x) \cdot Q(x) + R(x)}$, or equivalently $\boxed{\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}}$.

D is the divisor
Q is the quotient
R is the remainder

Here, $0 \le \deg(R) < \deg(D)$.

4–5. Examples Divide $3x^3 - 2x + 5$ by $x - 3$.

4. Long Division

$$
\begin{array}{r}
3x^2 + 9x + 25 \\
x-3 \,\overline{\big)\, 3x^3 + 0x^2 - 2x + 5} \\
\underline{3x^3 - 9x^2 \phantom{{}-2x+5}} \\
9x^2 - 2x + 5 \\
\underline{9x^2 - 27x \phantom{{}+5}} \\
25x + 5 \\
\underline{25x - 75} \\
80
\end{array}
$$

So

$$\boxed{\frac{3x^3 - 2x + 5}{x-3} = 3x^2 + 9x + 25 + \frac{80}{x-3}}$$

5. Synthetic Division

$$
\begin{array}{r|rrrr}
3 & 3 & 0 & -2 & 5 \\
 & \downarrow & 9 & 27 & 75 \\
\hline
 & 3 & 9 & 25 & \boxed{80}
\end{array}
$$

So $\boxed{3x^3 - 2x + 5 = (x-3)(3x^2 + 9x + 25) + 80}$

6. Example Use synthetic division to divide $5x^3 - 6x^2 - 28x + 8$ by $x + 2$.

$$
\begin{array}{r|rrrr}
-2 & 5 & -6 & -28 & 8 \\
 & \downarrow & -10 & 32 & -8 \\
\hline
 & 5 & -16 & 4 & \boxed{0}
\end{array}
$$

So, $\boxed{\frac{5x^3 - 6x^2 - 28x + 8}{x+2} = 5x^2 - 16x + 4}$

7. Remainder Theorem

Let P be a polynomial and $D(x) = x-k$, $k \in \mathbb{R}$.

Then $R(x) = R$ is a constant and

$$\boxed{P(k) = R}.$$

If $R=0$, then $x=k$ is a zero of P.

8. Example Use synthetic division to find $P(-2)$ where $P(x) = -x^4 + 3x^2 - 4x - 5$. Likewise, find $P(1)$.

$$\begin{array}{r|rrrrr} -2 & -1 & 0 & 3 & -4 & -5 \\ & \downarrow & 2 & -4 & 2 & 4 \\ \hline & -1 & 2 & -1 & -2 & \boxed{-1} \end{array}$$

so, $\boxed{P(-2) = -1}$ by the Remainder Thm.

9. Example Decide whether the given numbers are zeros of the polynomial.

(*a.*) 2; $P(x) = x^3 - 4x^2 + 9x - 10$

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 9 & -10 \\ & \downarrow & 2 & -4 & 10 \\ \hline & 1 & -2 & 5 & \boxed{0} \end{array}$$

So $x=2$ is a zero of P.

(*b.*) -2; $P(x) = \frac{3}{2}x^3 - x^2 + \frac{3}{2}x = \frac{1}{2}\left(3x^3 - 2x^2 + 3x\right)$

$$\begin{array}{r|rrrr} -2 & 3 & -2 & 3 & 0 \\ & \downarrow & -6 & 16 & -38 \\ \hline & 3 & -8 & 19 & \boxed{-38} \end{array}$$

So $x=-2$ is not a zero of P.

FYI,

$$P(x) = \frac{1}{2}\Big((x+2)(3x^2-8x+19) - 38 \Big)$$

$$= \frac{1}{2}(x+2)(3x^2-8x+19) - 19$$

so $\boxed{P(-2) = -19}$ NOT -38

10. Factor Theorem

Let P be a polynomial function and $D(x) = x - k$, $k \in \mathbb{R}$.
Then D is a factor of P if and only if $R = 0$.
Equivalently, D is a factor of P if and only if $P(k) = 0$.

11. Example Determine whether $D(x) = x+2$ is a factor of $P(x) = 4x^3 + 24x^2 + 48x + 32$. If it is, write P as a product $P(x) = D(x) \cdot Q(x)$.

$$\begin{array}{r|rrrr} -2 & 4 & 24 & 48 & 32 \\ & \downarrow & -8 & -32 & -32 \\ \hline & 4 & 16 & 16 & \boxed{0} \end{array}$$

Yes, $x+2$ is a factor!

$$\boxed{P(x) = (x+2)(4x^2 + 16x + 16)}$$

12. Example Divide $x^3 + \frac{5}{2}x^2 + x + 2$ by $2x^2 + x - 1$ and write the answer in both $P = D \cdot Q + R$ form, and $\frac{P}{D} = Q + \frac{R}{D}$.

$$\begin{array}{r|l} & \frac{1}{2}x + 1 \\ \hline 2x^2 + x - 1 & x^3 + \frac{5}{2}x^2 + x + 2 \\ & x^3 + \frac{1}{2}x^2 - \frac{1}{2}x \\ \hline & 2x^2 + \frac{3}{2}x + 2 \\ & 2x^2 + x - 1 \\ \hline & \frac{1}{2}x + 3 \end{array}$$

Remainder!

$P = D \cdot Q + R$:

$$x^3 + \frac{5}{2}x^2 + x + 2 = (2x^2 + x - 1)(\tfrac{1}{2}x + 1) + \tfrac{1}{2}x + 3$$

$\frac{P}{D} = Q + \frac{R}{D}$:

$$\frac{x^3 + \frac{5}{2}x^2 + x + 2}{2x^2 + x - 1} = \frac{1}{2}x + 1 + \frac{\frac{1}{2}x + 3}{2x^2 + x - 1}$$