13. The Fundamental Theorem of Algebra

Every polynomial of degree at least 1 has at least one complex root.

14. Number of Zeros Theorem

A function defined by a polynomial of degree n has at most n distinct complex roots.

15. Conjugate Zeros

If P is a polynomial w/ real coefficients, and if a+bi is a root of P, then the conjugate a-bi is also a root of P.

16. Example Given that $(1-i)$ is a zero of the polynomial function, find all complex zeros.

$$P(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$$

$(1+i)$ is also a root.

$(x-(1-i))(x-(1+i)) = x^2 - (1-i)x - (1+i)x + (1-i)(1+i) = x^2 - 2x + 2$

$$
\begin{array}{r}
x^2 - 5x + 6 \\
x^2 - 2x + 2 \,\big)\, \overline{x^4 - 7x^3 + 18x^2 - 22x + 12} \\
x^4 - 2x^3 + 2x^2 \qquad\qquad \\
\hline
-5x^3 + 16x^2 - 22x + 12 \\
-5x^3 + 10x^2 - 10x \qquad \\
\hline
6x^2 - 12x + 12 \\
6x^2 - 12x + 12 \\
\hline
0
\end{array}
$$

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2, 3$

So all roots of P are

$\boxed{1-i,\ 1+i,\ 2,\ 3}$

17. Example Find a polynomial function with real coefficients of least possible degree having a zero at 2 of multiplicity 3, at 0 of multiplicity 2, and a zero i of single multiplicity.

$$(x-2)^3(x-0)^2(x-i)(x+i)$$

$$(x-2)^3 x^2 (x^2+1)$$

$$(x^4+x^2)(x^3-6x^2+12x-8)$$

$$x^7-6x^6+12x^5-8x^4+x^5-6x^4+12x^3-8x^2$$

so, $\boxed{P(x)=x^7-6x^6+13x^5-14x^4+12x^3-8x^2}$

18. Rational Roots Theorem

Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ w/ $a_n \neq 0$ and $a_0 \neq 0$. If $\frac{p}{q}$ is rational number written in lowest terms, and if $\frac{p}{q}$ is a root of P, then p is a factor of a_0 and q is a factor of a_n.

* This reduces the number of possible roots that we need to check when trying to factor P into linear factors.

19. Example Find all rational zeros of $P(x) = 6x^3 - 5x^2 - 7x + 4$ and factor P.

Possible

p: $\pm 1, \pm 2, \pm 4$

q: $\pm 1, \pm 2, \pm 3, \pm 6$

Answers turn out to be:

$\boxed{\frac{1}{2}, -1, \frac{4}{3}}$

obtained by plugging in each possible root

$$\begin{array}{r|rrrr} -1 & 6 & -5 & -7 & 4 \\ & \downarrow & -6 & 11 & -4 \\ \hline & 6 & -11 & 4 & \boxed{0} \end{array}$$

$P(x)=(x+1)(6x^2-11x+4)$ ← last factor can be solved using q. formula.

$$x = \frac{11 \pm \sqrt{121-96}}{12} = \frac{11 \pm \sqrt{25}}{12} = \frac{11 \pm 5}{12} = \frac{16}{12}, \frac{6}{12}$$

so $\boxed{x = \frac{4}{3}, \frac{1}{2}}$

20. Example Find all rational zeros and factor:

$$P(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$$

$$\begin{array}{r|rrrrr} 1 & 6 & 7 & -12 & -3 & 2 \\ & \downarrow & 6 & 13 & 1 & -2 \\ \hline -2 & 6 & 13 & 1 & -2 & \boxed{0} \\ & \downarrow & -12 & -2 & 2 & \\ \hline & 6 & 1 & -1 & \boxed{0} & \end{array}$$

So $P(x) = (x-1)(x+2)(6x^2 + x - 1)$

$x = \dfrac{-1 \pm \sqrt{1+24}}{12} = \dfrac{-1 \pm 5}{12} = \dfrac{-6}{12}, \dfrac{4}{12}$

$x = -\frac{1}{2}, \frac{1}{3}$

so $\boxed{P(x) = 6(x-1)(x+2)(x+\tfrac{1}{2})(x-\tfrac{1}{3})}$

21. Descartes's Rule of Signs

Let P be a polynomial w/ real coefficients and a nonzero constant term, w/ terms in descending powers of x.

1.) The number of positive real zeros is either equal to the number of sign changes, or is less than this number by a positive even integer.

2.) The number of negative real zeros is either equal to the number of sign changes in $P(-x)$, or differs by an even positive integer.

22. Example Determine the possible number of positive real roots and negative real roots of P.

(*a.*) $P(x) = x^4 - 6x^3 + 8x^2 + 2x - 1$

Positive: 3 or 1

Negative: $P(-x) = x^4 + 6x^3 + 8x^2 - 2x - 1$

: So, 1.

(*b.*) $P(x) = x^5 - 3x^4 + 2x^2 + x - 1$

Positive: 3 or 1

Negative: $P(-x) = -x^5 - 3x^4 + 2x^2 - x - 1$

: 2 or 0

23. Boundedness Theorem

Let P be a polynomial of degree $n \geq 1$ and a positive leading coefficient. Suppose P is divided synthetically by $(x-c)$.

1.) If $c>0$ and all numbers in the bottom row of the synthetic division are nonnegative, then P has no root greater than c.

2.) If $c<0$ and the numbers in the bottom row of the synthetic division alternate signs, then P has no root less than c.

24. Example Show that the polynomial function

$$P(x) = 2x^4 - 5x^3 + 3x + 1$$

satisfies the following conditions. (*a.*) P has no real zero larger than 3, and (*b.*) P has no real zero less than -1.

a.)

$$\begin{array}{r|rrrrr} 3 & 2 & -5 & 0 & 3 & 1 \\ & \downarrow & 6 & 3 & 9 & 36 \\ \hline & 2 & 1 & 3 & 12 & \boxed{37} \end{array} \leftarrow$$

All positive, so roots must be less than 3.

(*) Notice 3 is itself <u>not</u> a root of P.

b.)

$$\begin{array}{r|rrrrr} -1 & 2 & -5 & 0 & 3 & 1 \\ & \downarrow & -2 & 7 & -7 & 4 \\ \hline & 2 & -7 & 7 & -4 & \boxed{5} \end{array} \leftarrow$$

Signs alternate, so roots must be greater than -1.