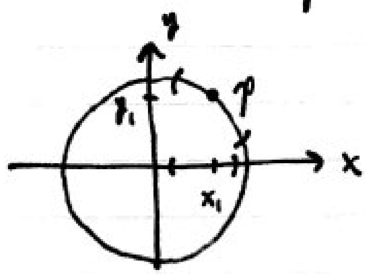
Implicit differentiation

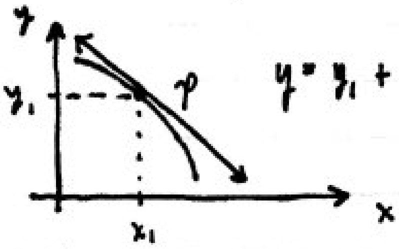
Idea: we want to find an equation of the tangent line to a curve that is not defined by a function globally.

Example. The unit circle $x^2+y^2=1$



Near any point p on the circle except the two x-intercepts, the portion of the curve around p may be regarded as an implicit function $y=y(x)$.

The tangent line at p looks like



$$y=y_1+\left.\frac{dy}{dx}\right|_p (x-x_1)$$

Here $m=\left.\frac{dy}{dx}\right|_p$ is the slope of the tangent line at p.

To compute $\left.\frac{dy}{dx}\right|_p$ we take $\frac{d}{dx}$ of the relation

$$\frac{d}{dx}\left[x^2+y^2=1\right]$$

$$\Rightarrow \quad \frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=\frac{d}{dx}\left[1\right]$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \qquad (*)$$

where $\frac{d}{dx}[y^2] = 2y\frac{dy}{dx}$ because we are regarding $y = y(x)$ as a function of x near p, so its derivative requires a chain rule.

Now we rearrange equation $(*)$ to solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

This means that the slope of the tangent line to the circle $x^2+y^2=1$ at the point $p(x_1,y_1)$ is given by

$$\left.\frac{dy}{dx}\right|_p = \frac{-x_1}{y_1}.$$

Thinking back to our original assumptions, this actually makes sense at the x-intercepts as well: the tangent lines to the curve at those points is vertical.

<u>Examples</u>. Find an equation of the tangent line to the curve at the given point:

a.) $x^2+y^2=25$ at $(3,-4)$

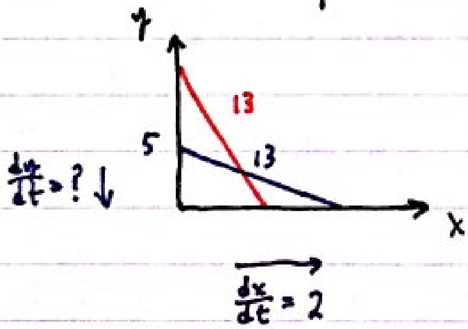
b.) $x^3+y^3=6xy$ at — Folium of Descartes

c.) $(x^2+y^2)^3 = 4x^2y^2$ at — Four-Leaved Rose

Related Rates

This is merely an application of implicit differentiation. Suppose two quantities $A = A(t)$ and $B = B(t)$ are changing in time and are related by an equation. Then their rates of change are also related by taking $\frac{d}{dt}$ of the equation relating the quantities.

Example. A 13 foot tall ladder is leaning against a wall. If the bottom of the ladder is sliding away from the wall at a rate of 2 feet per minute, how fast is the top of the ladder sliding down the wall when the top is 5 feet from the ground?



$$x^2 + y^2 = 13^2$$

$$x^2 + 5^2 = 169$$

$$x^2 = 144$$

$$x = 12 \text{ feet}$$

$$\frac{d}{dt}\left[x^2 + y^2 = 169\right]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\cancel{\cancel{2}(12)(\cancel{5}) + \cancel{2}(\cancel{5})\frac{dy}{dt} = 0}$$

$$\cancel{12 + \frac{dy}{dt} = 0}$$

$$\cancel{\boxed{\frac{dy}{dt} = -12 \text{ ft/minute.}}}$$

↘

$$\cancel{2}(12)(2) + \cancel{2}(5)\frac{dy}{dt} = 0$$

$$5\frac{dy}{dt} = -24$$

$$\frac{dy}{dt} = \frac{-24}{5} \text{ ft/min}$$

$$= \boxed{-4.8 \text{ ft/min}}$$

More related rates examples: handout and GPs.

Next: Some Important Theorems

Skip until pg. 6

<u>Rolle's Theorem</u>
Let f be a differentiable function on a closed interval $[a,b]$ satisfying $f(a)=f(b)$. Then there exists a number $a<c<b$ satisfying $f'(c)=0$.

To prove this, we need a couple of other fundamental results.

<u>Theorem</u>. If f is differentiable on an open interval (a,b), then f is continuous on (a,b).

<u>Caution</u>: This is a one-way theorem! Differentiable implies continuous, but <u>not</u> vice versa. Can you think of an example.

<u>Proof</u>. We only need to prove this at a point $x=c$, $a<c<b$.

Since f is differentiable at $x=c$, then

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \qquad \text{exists.}$$

Consider $\lim_{x\to c}\left[f(x)-f(c)\right]=\lim_{x\to c}(x-c)\frac{f(x)-f(c)}{x-c}$

$$=\lim_{x\to c}(x-c)\cdot\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$=f'(c)\cdot\lim_{x\to c}x-c$$

$$=0.$$

Therefore $\lim_{x\to c}(f(x)-f(c)) = 0$.

On the other hand,

$$\lim_{x\to c}(f(x)-f(c)) = \lim_{x\to c} f(x) - f(c)$$

Putting these together, we obtain

$$\lim_{x\to c} f(x) - f(c) = 0$$

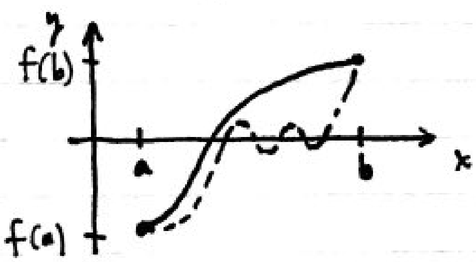
or $$\lim_{x\to c} f(x) = f(c).$$

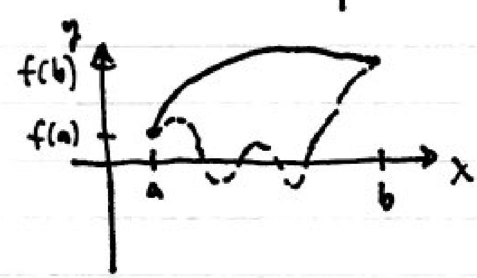
Therefore f is continuous at c. ■

Next, we need the

<u>Intermediate Value Theorem</u> (Balzano's Theorem)
Let f be a continuous function on a closed interval $[a,b]$, ~~Then~~ and suppose $f(a)$ and $f(b)$ are nonzero and have opposite signs. Then f has a real root in (a,b).

We won't prove this, but we will draw a picture:





f could have many roots in (a,b), but the theorem <u>guarantees</u> at least one.

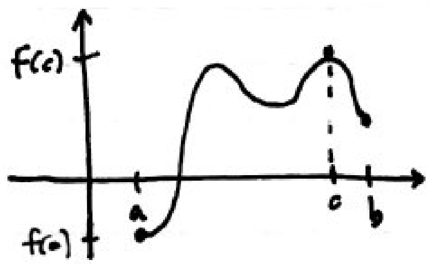
The theorem is inconclusive when $f(a)$ and $f(b)$ have the same sign.

Now, the

Extreme Value Theorem

Let f be a continuous function on the closed interval $[a,b]$. Then f attains an absolute maximum and absolute minimum value on $[a,b]$.

Again, we'll skip the proof but draw a picture.



for this function, the minimum is attained at an endpoint and the maximum at a critical point.

If the max/min is attained at an interior point, then that will always correspond to a critical number of the function: either $f'(c) = 0$ or $f'(c) =$ undefined.

We're now ready to prove Rolle's Theorem. Recall, it says:

If f is a differentiable function on (a,b), continuous on $[a,b]$, and $f(a) = f(b)$, then there exists a number c in (a,b) such that $f'(c) = 0$.

(This statement is equivalent — but better — than the one on page 4. It explains what we mean by differentiable on a closed interval.)

To prove the theorem, we put $f(a) = f(b) = k$ and define a new function $g(x) = f(x) - k$. So $g(a) = g(b) = 0$.

<u>Proof</u>. If $g(x)=0$ is constant, then $g'(x)=0$ is constant and every point in (a,b) satisfies the theorem.

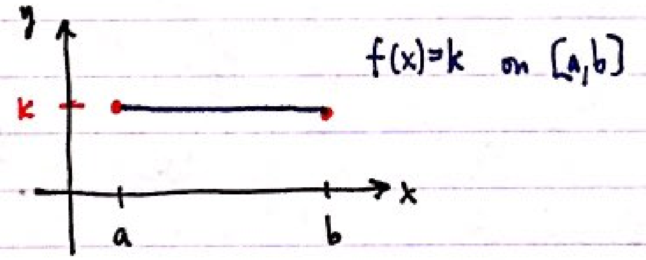
If g varies on (a,b), then there are points in (a,b) where either $g(x)>0$ or $g(x)<0$.

If there are points where $g(x)>0$, then g will attain its maximum in (a,b) by the Extreme Value Theorem. Since g is differentiable, then $g'(x)$ exists for all x in (a,b), and therefore $g'(c)=0$ at the maximum value.

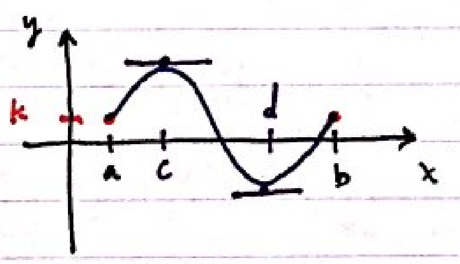
<u>Exercise</u>. Write the proof for $g(x)<0$.

This proves the theorem. ∎

Pictures of Rolle's Theorem:



In this case, $f'(x)=0$ at every point in (a,b).



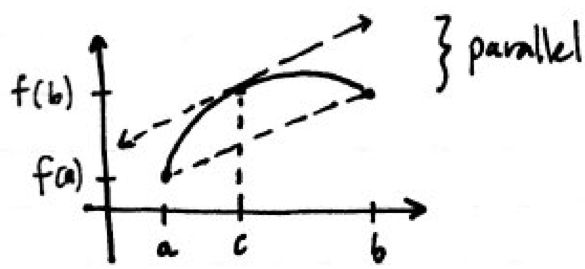
If $f(x)$ is not constant on $[a,b]$, then the graph of f must have at least one turning point in (a,b). At this point $f'(c)=0$.

Finally, we generalize Rolle's Theorem:

Mean Value Theorem.
Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Remarks. This says that there is a point in (a,b) where the slope of the tangent line is the same as the average (mean) rate of change of the function on $[a,b]$.
i.e., the tangent line is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$.



Again, the theorem asserts that there is at least one such number in the interval (a,b). There could be more.

Proof. The equation of the secant line through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Define a new function

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Notice that $g(a) = g(b) = 0$, g is continuous on $[a,b]$ because f is, and g is differentiable on (a,b) because f is. Then Rolle's Theorem applies and says there is a c in (a,b) such that

$$g'(c) = 0.$$

But $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$, so this implies that

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \blacksquare$$

Some applications, then Good Problems! (Hand out)