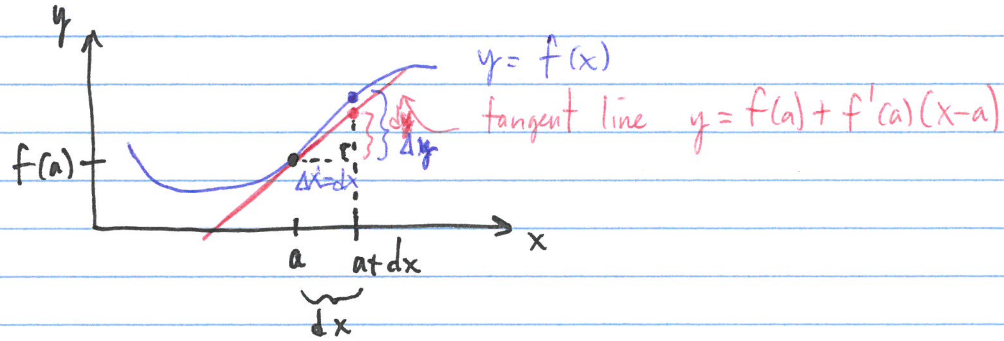
3.10 Linear Approximations and Differentials



The function determined by the tangent line to f at p(a, f(a)) is called the <u>linearization of f at p</u>, denoted

$$\boxed{y = L_pf(x) = f(a) + f'(a)(x-a)}$$ Linear Approximation

Then $f(x) \approx L_pf(x)$ for all x "near" a, but $L_pf(x)$ is usually much easier to compute.

If we write $f(a+dx) \approx L_pf(a+dx) = f(a) + f'(a)(a+dx-a)$

$$= f(a) + f'(a)\,dx$$

Recall that $\frac{dy}{dx} = f'(a)$, so $dy = f'(a)\,dx$.

This is called the <u>differential</u> of $y = f(x)$ at $x = a$. It measures the change in the tangent line wrt to x, as opposed to Δy which measures the change in the actual function.

Subbing in, we can now write

$$\boxed{f(a+dx) \approx f(a) + dy}$$ Differential

This can also be used to approximate values of $y = f(x)$ near $x = a$.

<u>Lots of examples.</u>

<u>Ex.</u> $f(x) = \sqrt{x+3}$ Find $L_p f$ at $p(1,2)$.

$f'(x) = \dfrac{1}{2\sqrt{x+3}}$ $\qquad f'(1) = \dfrac{1}{4}$

$$L_p f(x) = 2 + \tfrac{1}{4}(x-1) = \boxed{\tfrac{1}{4}x + \tfrac{7}{4} = L_p f(x)}$$

Use this to approximate the value of $\sqrt{3.98}$ and $\sqrt{4.05}$

$$\sqrt{3.98} = \sqrt{0.98+3} \approx \frac{1}{4}\cdot\frac{98}{100} + \frac{7}{4} = \frac{49}{200} + \frac{7}{4} = \frac{49+7\cdot 50}{200}$$

$$= \frac{49+350}{200} = \frac{399}{200} = \underline{1.995}$$

<u>actual</u>: 1.99499373...

For $\sqrt{4.05}$ consider $dx = 0.05 = \frac{1}{20}$

then $dy = f'(1)\,dx = \dfrac{1}{4}\cdot\dfrac{1}{20} = \dfrac{1}{80}$

so $\sqrt{4.05} \approx 2 + \frac{1}{80} = \underline{2.0125}$

<u>actual</u>: 2.01246117...

<u>Ex.</u> $y = x^3 + x^2 - 2x + 1$ $\qquad y(2) = 8+4-4+1 = 9$

Compare Δy and dy as $x: 2 \mapsto 2.01$ and $x: 2 \mapsto 2.05$

$y' = 3x^2 + 2x - 2$ $\qquad y'(2) = 12+4-2 = 14$

$dy = 14\,dx$

for $dx = 0.05$ $\quad dy = \frac{14}{20} = \frac{7}{10} = 0.7$

$dx = 0.01$ $\quad dy = \frac{14}{100} = 0.14$

at 2.05: $\qquad$ <u>phew</u>.

$$\Delta y = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 - 9$$
$$= \cancel{8} + 3\cdot 4\cdot\left(\tfrac{1}{20}\right) + 3\cdot 2\cdot\left(\tfrac{1}{400}\right) + \tfrac{1}{8000} + 4 + \tfrac{4}{20} + \tfrac{1}{400} - \cancel{4} - \tfrac{2}{20} - \cancel{8}$$
$$= \frac{14}{20} + \frac{6}{400} + \frac{1}{8000} = \frac{5600+120+1}{8000}$$
$$= \frac{5721}{8000} = 0.715125$$

at 2.01 : $\Delta y = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 - 9$

$$= \cancel{8} + 3\cdot4\cdot\frac{1}{100} + 3\cdot2\cdot\frac{1}{1000} + \frac{1}{10000} + \cancel{4} + 2\cdot2\cdot\frac{1}{100} + \frac{1}{1000} \cancel{-4} - 2\frac{1}{100} \cancel{+1} \cancel{-9}$$

$$= \frac{1200}{10000} + \frac{60}{10000} + \frac{1}{10000} + \frac{400}{10000} + \frac{10}{10000} - \frac{200}{10000}$$

$$= \frac{1471}{10000} = 0.1471$$

In conclusion :

$2 \to 2.01$: $dy = 0.14$, $\Delta y = 0.1471$

$2 \to 2.05$: $dy = 0.7$, $\Delta y = 0.715725$

The differentials are not perfect, but they take <u>much</u> less work!

<u>Ex.</u> The radius of a sphere was measured to be 21 cm w/ a possible error of at most 0.05 cm.

~~What is~~ Estimate the maximum error in using this value of the radius to compute the volume of the sphere?

$V = \frac{4}{3}\pi r^3 \qquad dV = 4\pi r^2\, dr$

If $dr = 0.05$, then $dV = 4\pi(21)^2\left(\frac{1}{20}\right) = \frac{21^2\pi}{5} = \frac{441\pi}{5} \approx 277 \text{ cm}^3$

This looks like a lot, but consider the relative error

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = 3\,\frac{dr}{r} = \frac{3}{20\cdot21} = \frac{3}{420} \approx 0.007$$

So the error is at most 0.7% of the total volume.

<u>Ex.</u> Find the differentials dy for

a.) $y = \cos(\pi x)$

b.) $y = e^{1/x}$

c.) $y = \frac{1 - x}{1+x}$

Ex. Consider $f(x) = (x-1)^2$, $g(x) = e^{-2x}$, $h(x) = 1 + \ln(1-2x)$

Calculate the linearization for each one at $a = 0$.

$f'(x) = 2(x-1)$ $\quad f(0) = 1$ $\quad f'(0) = -2$

$$L_a f(x) = 1 - 2(x-1) = \boxed{-2x + 3}$$

$g'(x) = -2e^{-2x}$ $\quad g(0) = 1$ $\quad g'(0) = -2$

$$L_a g(x) = 1 - 2(x-1) = \boxed{-2x + 3}$$

$h'(x) = \frac{-2}{1-2x}$ $\quad h(0) = 1$ $\quad h'(0) = -2$

$$L_a h(x) = 1 - 2(x-1) = \boxed{-2x + 3}$$

* Graph all 4. Which one is best approximated by L?

Ex. Use differentials to estimate the amount of paint needed to apply a 0.05 cm thick coat to a hemispherical dome w/ diameter 50 cm.

$SA = 4\pi r^2$

$$d(SA) = 8\pi r\, dr = 4\pi d \cdot dr$$
$$= 4\pi(50) \cdot \frac{1}{20} = 10\pi \approx 31.4 \text{ cm}^2$$