M242 8/31

Ch 2. Derivatives

§2.1. Let f be a function, $x=a$ a pt in the domain.

The **derivative** of f at $x=a$, if it exists, is the slope of the tangent line to the graph of f at $x=a$.

The derivative of f at a, $f'(a)$, is the instantaneous rate of change of f at a.

$$f'(a) = \dot{f}(a) = \left.\frac{dy}{dx}\right|_{x=a} = \frac{df}{dx}(a)$$

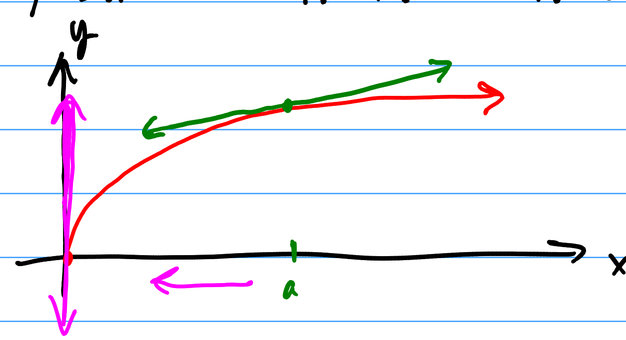
Q. What could go wrong?

1. $f(a) =$ undef. (hole) No tangent line at a.

Tan. line: $y = f(a) + f'(a)(x-a)$

2. Tangent line exists, but is vertical. $m =$ undef.

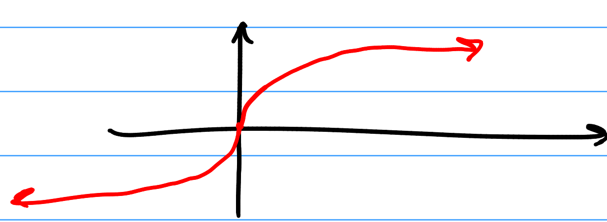
Ex.



$y = \sqrt{x}$

$y' = \frac{1}{2\sqrt{x}}$, $x \neq 0$

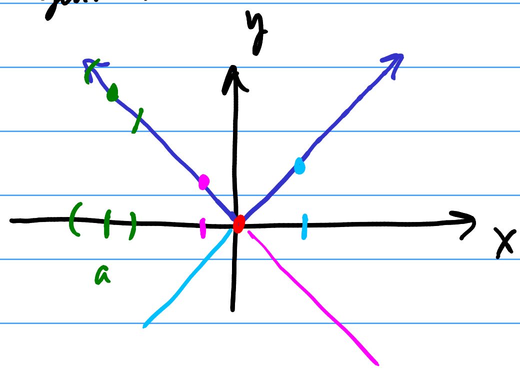
or



$y = \sqrt[3]{x}$

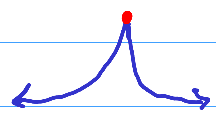
3. No unique tangent line

Ex. $y = |x|$



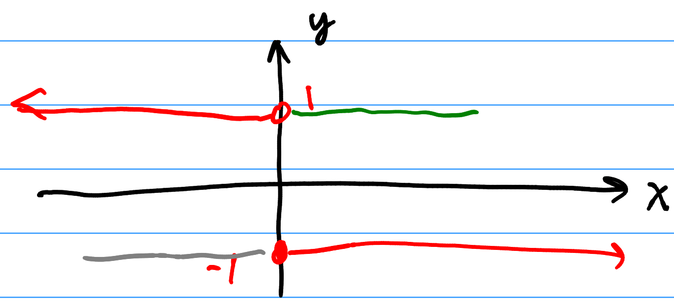
corner.

$$\left.\begin{aligned} \lim_{x\to 0^-} \frac{|x|-|0|}{x-0} &= -1 \\ \lim_{x\to 0^+} \frac{|x|-|0|}{x-0} &= +1 \end{aligned}\right\} \neq$$

cusp. 

vertical tan. line

Ex.



$$\left.\begin{aligned} \lim_{x\to 0^-} \dots &= 0 \\ \lim_{x\to 0^+} \dots &= 0 \end{aligned}\right\} =$$

BUT there are 2 tan. lines.

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§2.2 Derivatives as functions

at a point: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

We can allow a to vary in the domain of f, and thus regard $f'(a)$ as a function.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$ function!

domain of f' is all points in dom(f) for which the limit exists.

Ex. $f(x) = \sqrt{x}$ $\qquad$ $\text{dom}(f) = [0, \infty)$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad$ $\text{dom}(f') = (0, \infty)$

Ex. $f(x) = x^3 + x$ $\qquad$ find a formula for $f'(x)$.

$\text{dom}(f) = \mathbb{R}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 + (x+h) - (x^3 + x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{x} + h - \cancel{x^3} - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 + 1) = 3x^2 + 1$$

1

1 1

1 2 1

$\boxed{1 \ 3 \ 3 \ 1}$

$$\left.\begin{array}{l} f(x) = x^3 + x \\ f'(x) = 3x^2 + 1 \end{array}\right\} \text{ domains are both } \mathbb{R}.$$

Ex. find a formula for the tan. line to $f(x) = x^3 + x$ at $x = 2$.

$y - y_1 = m(x - x_1) \leftarrow m = \frac{y - y_1}{x - x_1} \leftarrow$

$y = y_1 + m(x - x_1)$

$$y = f(a) + f'(a)(x - a)$$

✓ $a = 2$

✓ $f(a) = f(2) = 2^3 + 2 = 10$

✓ $f'(a) = f'(2) = 3(2^2) + 1 = 13$

$$y = 10 + 13(x - 2)$$

$$\boxed{y = 13x - 16} \ \circledast$$

Ex. $g(x) = \frac{2x-1}{x+2}$ dom(g): $x \neq -2$

Find $g'(x)$.

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x-a} = \lim_{x \to a} \frac{\frac{2x-1}{x+2} - \frac{2a-1}{a+2}}{x-a}$$

$$= \lim_{x \to a} \frac{\frac{(a+2)}{(a+2)} \frac{(2x-1)}{(x+2)} - \frac{(x+2)}{(x+2)} \frac{(2a-1)}{(a+2)}}{x-a}$$

$$= \lim_{x \to a} \frac{\cancel{2ax} + 4x - a \cancel{-2} - [\cancel{2ax} + 4a - x \cancel{-2}]}{(x+2)(a+2)(x-a)}$$

$$= \lim_{x \to a} \frac{4x - a - 4a + x}{(x+2)(a+2)(x-a)} = \lim_{x \to a} \frac{5\cancel{(x-a)}}{(x+2)(a+2)\cancel{(x-a)}}$$

$$= \lim_{x \to a} \frac{5}{(x+2)(a+2)} = \boxed{\frac{5}{(a+2)^2} = f'(a)}$$

$f(x) = \frac{2x-1}{x+2}$ $f'(x) = \frac{5}{(x+2)^2}$ dom. of both : $x \neq -2$