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Ch 2 - Derivatives

$$y=f(x), \qquad f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$f'(a)$ is the slope of the tangent line to the graph of $y=f(x)$ at $x=a$:

$$y=f(a)+f'(a)(x-a)$$

$f'(a)$ is the instantaneous rate of change of f at $x=a$.

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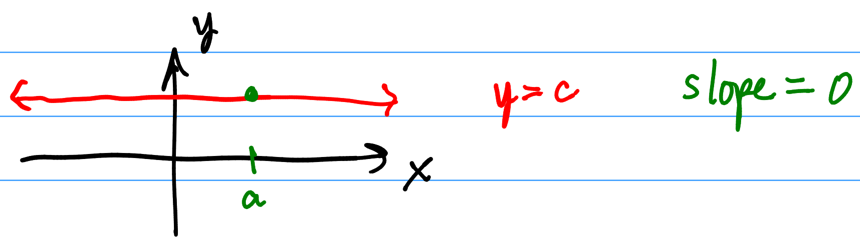
§2.3 - Derivative Rules

Ex. $f(x)=c$ for some constant c.



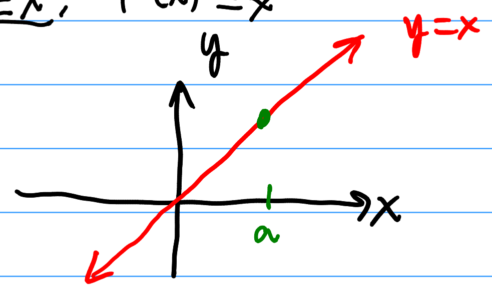
$$\boxed{\frac{d}{dx}[c]=0}$$ (*)

or $(c)'=0$

Proof. $\left.\begin{array}{l} f(x)=c \\ f(a)=c \end{array}\right\}$ $\quad f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{c-c}{x-a}=\lim_{x\to a}\frac{0}{x-a}$

$=0$ ∎

Ex. $f(x) = x$



$$\boxed{\frac{d}{dx}[x] = 1} \quad \circledast$$

$$(x)' = 1$$

Proof: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\cancel{x-a}}{\cancel{x-a}} = \lim_{x \to a} 1 = 1.$ ■

$f(x) = x$

$f(a) = a$

Ex. $f(x) = x^2$

$f'(x) = 2x \leftarrow ?$

$$f'(a) = \lim_{x \to a} \left[\frac{x^2 - a^2}{x - a} \right]$$

$\uparrow$

$x + a$

$$\begin{array}{c|ccc} & x^2 & x & \# \\ a & 1 & 0 & -a^2 \\ & \downarrow & a & a^2 \\ \hline & 1 & a & \boxed{0} \\ & x & \# & \end{array}$$

$$= \lim_{x \to a} (x + a) = a + a = 2a$$

$f'(a) = 2a$ or $f'(x) = 2x$

Ex. $f(x) = x^3$

$f'(x) = 3x^2$

$$f'(a) = \lim_{x \to a} \frac{x^3 - a^3}{x - a}$$

$$\begin{array}{c|cccc} & x^3 & x^2 & x & \# \\ a & 1 & 0 & 0 & -a^3 \\ & \downarrow & a & a^2 & a^3 \\ \hline & 1 & a & a^2 & \boxed{0} \end{array}$$

$$= \lim_{x \to a} (x^2 + ax + a^2)$$

$$= a^2 + a \cdot a + a^2 = 3a^2$$

$f'(a) = 3a^2$ or $f'(x) = 3x^2$

Rule: if $f(x) = x^n$ for any n, then

$$\boxed{\frac{d}{dx}[x^n] = n\,x^{n-1}} \quad (*) \qquad \text{Power Rule}$$

Ex. $f(x) = x^0 = 1$ $\qquad f'(x) = 0\,x^{0-1} = 0 \cdot x^{-1} = 0$

$f'(x) = 0$

Ex. $f(x) = \sqrt{x} \longrightarrow f(x) = x^{1/2}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f'(x) = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$.

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Ex. $g(x) = \frac{1}{x} \rightarrow g(x) = x^{-1}$

$g'(x) = \frac{-1}{x^2}$ $\qquad g'(x) = -1 \cdot x^{-1-1} = -1 \cdot x^{-2} = \frac{-1}{x^2}$

Ex.

$$\begin{cases} \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = f'(x) + g'(x). \\ \frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \end{cases}$$

RE. Prove it.

Ex. $\frac{d}{dx}[k \cdot f(x)] = k \cdot f'(x)$

Ex. $\frac{d}{dx}\left[4x^3 - 3x + \frac{2}{x}\right] = (4x^3)' - (3x)' + (2/x)'$

$= 4(x^3)' - 3(x)' + 2(1/x)'$

$= 4 \cdot 3x^2 - 3 \cdot 1 + 2(-1/x^2)$

$f'(x) = 12x^2 - 3 - 2/x^2$

Ex. $\frac{d}{dx}[f(x) \cdot g(x)]$

$f(x) = x$ $\qquad f'(x) = 1$ $\qquad (f(x)g(x))' = (x^3)' = 3x^2$

$g(x) = x^2$ $\qquad g'(x) = 2x$

$\underbrace{(f(x)g(x))' \stackrel{?}{=} f'(x)g'(x)\ ?}_{3x^2 \neq 1 \cdot 2x}$ WRONG

Limit Def'n:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \underbrace{\lim_{h\to 0} f(x+h)}_{f(x)} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} + \underbrace{g(x)}_{g(x)} \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)}$$

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$$\boxed{\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)} \quad (*)$$

Product Rule: $(uv)' = u'v + uv'$

Ex. $f(x) = x$ $\qquad (fg)' = (x^3)' = 3x^2$

$g(x) = x^2$

$$\left.\begin{array}{l} f'(x) = 1 \\ g'(x) = 2x \end{array}\right\} (fg)' = f'g + g'f = 1 \cdot x^2 + 2x \cdot x$$

$$= x^2 + 2x^2$$

$$= 3x^2$$

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