M242

Ex. $F(x) = (6x^3)(7x^4)$ $\quad F(x) = 42x^7$

$F'(x) = ?$ $\quad F'(x) = 42 \cdot 7x^6 = 294x^6$

$$F = u(x)v(x) \Rightarrow \boxed{F'(x) = u'(x)v(x) + u(x)v'(x)} \quad (*)$$

$u = 6x^3 \quad u' = 18x^2$

$v = 7x^4 \quad v' = 28x^3$

$$F'(x) = (6x^3)(28x^3) + (7x^4)(18x^2)$$
$$= 168x^6 + 126x^6$$
$$= 294x^6$$

Quotient Rule: f, g are differentiable (f', g' exist), and $g(x) \neq 0$.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \qquad \text{let } F(x) = \frac{f(x)}{g(x)}$$

The limit definition of $F'(x)$ is

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)}\frac{g(x)}{g(x)} - \frac{f(x)}{g(x)}\frac{g(x+h)}{g(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{g(x)\left[f(x+h) - f(x)\right]}{g(x)g(x+h)\,h} - \lim_{h\to 0} \frac{f(x)\left[g(x+h) - g(x)\right]}{g(x)g(x+h)\,h}$$

$$= \frac{g(x)\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) - f(x)\left(\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\right)}{g(x)^2}$$

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$ ⊛

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$\left(\frac{hi}{lo}\right)' = \frac{lo \cdot d\text{-}hi \ \text{minus} \ hi \ d\text{-}lo}{lo \cdot lo}$$

Ex. $y = \dfrac{x^2+x-2}{x^3+6}$

$u = x^2+x-2 \qquad u' = 2x+1$

$v = x^3+6 \qquad v' = 3x^2$

$$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2} = \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4+x^3+12x+6 - (3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$\boxed{\left(\frac{x^2+x-2}{x^3+6}\right)' = \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}}$$

Ex. $F(x) = \dfrac{3x^2+2\sqrt{x}}{x} \qquad x>0$

$u = 3x^2+2\sqrt{x} \qquad u' = 6x + 2\left(\frac{1}{2\sqrt{x}}\right)$

$v = x \qquad v' = 1$

$$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2} = \frac{x\left(6x+\frac{1}{\sqrt{x}}\right) - (3x^2+2\sqrt{x})(1)}{x^2}$$

$$= \frac{6x^2 + \sqrt{x} - 3x^2 - 2\sqrt{x}}{x^2}$$

$\frac{x}{\sqrt{x}} \cdot \frac{(\sqrt{x})^2}{\sqrt{x}} = \sqrt{x}$

$$F'(x) = \frac{3x^2 - \sqrt{x}}{x^2} = \frac{3x^2}{x^2} - \frac{\sqrt{x}}{x^2} = 3 - x^{-3/2}$$

OR

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x} = \frac{3x^2}{x} + \frac{2\sqrt{x}}{x} \qquad x > 0$$

$$= 3x + 2x^{-1/2}$$

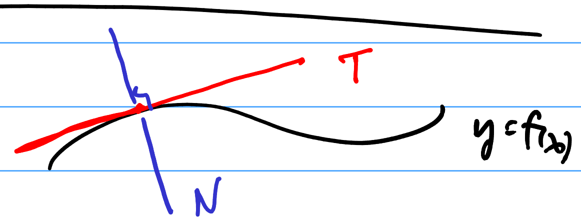
$$F'(x) = 3 + 2(-\tfrac{1}{2})x^{-3/2} = 3 - x^{-3/2}$$

Ex. $\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = f'(1) = 1000$

$f(x) = x^{1000} \qquad f(1) = 1^{1000} = 1$

$f'(x) = 1000x^{999} \qquad f'(1) = 1000 \cdot 1^{999} = 1000$

Ex. $y = \frac{\sqrt{x}}{1 + x^2} \qquad P(1, \tfrac{1}{2})$



Find eqns for the tangent line and normal line to $y = f(x)$ at P.

tan: $y = f(1) + f'(1)(x - 1)$

normal: $y = f(1) - \frac{1}{f'(1)}(x - 1)$

$$y = \frac{\sqrt{x}}{1 + x^2} \qquad y' = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} = \left.\frac{(1+x^2)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(2x)}{(1+x^2)^2}\right|_{x=1}$$

$u = \sqrt{x} \qquad u' = 1/2\sqrt{x}$

$v = 1 + x^2 \qquad v' = 2x$

$$= \frac{2\left(\frac{1}{2}\right) - 1(2)}{2^2} = \frac{-1}{4} = f'(1)$$

$$\frac{-1}{f'(1)} = 4$$

$$\begin{cases} T: \quad y = \frac{1}{2} - \frac{1}{4}(x-1) \\ N: \quad y = \frac{1}{2} + 4(x-1) \end{cases}$$

Think about $(\cos x)' = \lim_{h \to 0} \cdots$?