M242 9/6/18

§2.4 – Trig Derivatives

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h\to 0} \frac{\sin(h)\cos(x)}{h}$$

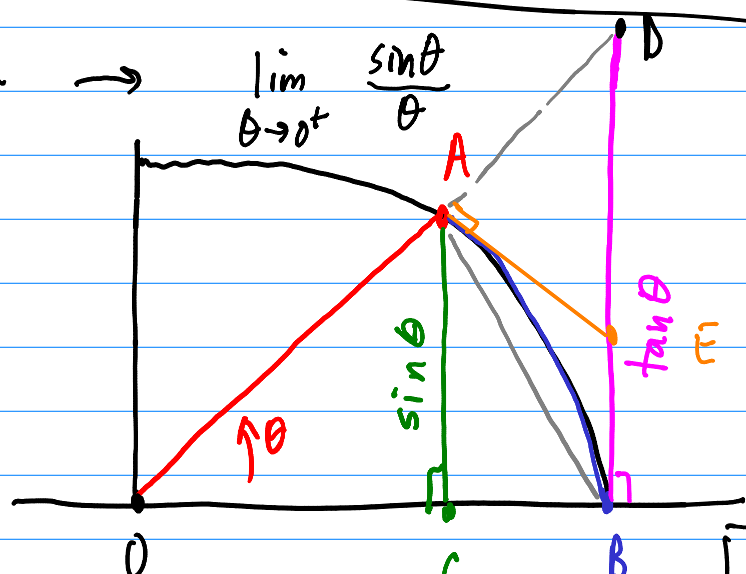
$$= \sin(x) \underbrace{\lim_{h\to 0} \frac{\cos(h) - 1}{h}} + \cos(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}$$

$$(\sin x)' = \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$\boxed{\frac{d}{dx}\left[\sin(x)\right] = \cos(x)} \quad (*)$$

Aside: $\lim_{h\to 0} \frac{\sin(h)}{h} \to \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}$

$x^2 + y^2 = 1$



$\theta = \text{arc}(AB)$

$\sin\theta = \overline{AC} < \overline{AB} < \text{arc}(AB) = \theta$

$\theta = \text{arc}(AB) < \overline{AE} + \overline{EB} < \overline{DE} + \overline{EB} = \tan\theta$

$\sin\theta < \theta \rightarrow \boxed{\frac{\sin\theta}{\theta} < 1}$

$$\theta < \frac{\sin\theta}{\cos\theta} \rightarrow \boxed{\cos\theta < \frac{\sin\theta}{\theta}}$$

Taking it all together,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\lim_{\theta\to 0^+} \cos\theta = \cos(0) = 1 \qquad \lim_{\theta\to 0^+} 1 = 1$$

By the squeeze theorem,

$$\boxed{\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1}$$

The argument for $\theta \to 0^-$ is identical, so

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$

Now,

$$\lim_{h\to 0} \frac{(\cos(h)-1)}{h}\frac{(\cos(h)+1)}{(\cos(h)+1)} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h+1)}$$

$$= -\lim_{h\to 0} \frac{\sin(h)}{h} \cdot \lim_{h\to 0} \frac{\sin(h)}{(\cos(h)+1)}$$

$$\left\{ \begin{aligned} \cos^2 h + \sin^2 h &= 1 \\ \cos^2 h - 1 &= -\sin^2 h \end{aligned} \right\}$$

$$= -1 \cdot \frac{0}{2} = 0$$

$$\boxed{\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0}$$

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$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= 0 - \sin(x) \cdot 1$$

$$= -\sin(x)$$

$$\boxed{\frac{d}{dx}[\cos(x)] = -\sin(x)} \quad *$$

Ex. $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x (\sin x)' - \sin x (\cos x)'}{\cos^2 x}$

$$= \frac{\cos x (\cos x) - \sin x (-\sin x)}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}[\tan x] = \sec^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$$

$$= \sec^2 x$$

Ex. $\frac{d}{dx}[\sec x] = \left(\frac{1}{\cos x}\right)' = \frac{\cos x (1)' - 1 \cdot (\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$\boxed{\frac{d}{dx}[\sec x] = \sec x \tan x}$$

$\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$