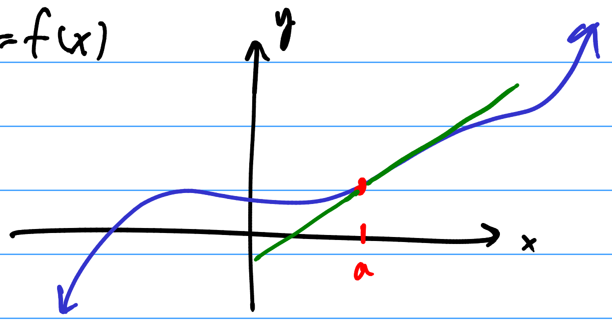
M242 9/11/18

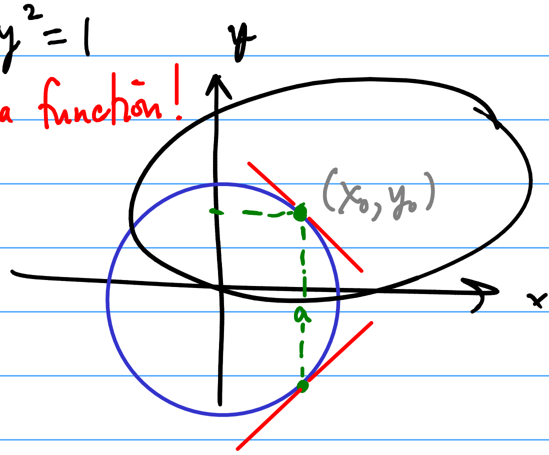
§2.6 ~ Implicit Differentiation

$y = f(x)$



$x^2 + y^2 = 1$

Not a function!



Ex. $x^2 + y^2 = 1$, $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ Find eq'n of tan. line

$$y = y_0 + \frac{dy}{dx}(x - x_0) = \frac{\sqrt{3}}{2} + \frac{dy}{dx}\left(x - \tfrac{1}{2}\right)$$

$$x^2 + y^2 = 1$$

$$\sqrt{y^2} = \sqrt{1 - x^2}$$

$$y = \pm\sqrt{1 - x^2}$$

$$\boxed{y = \sqrt{1 - x^2}}$$

Compute $\frac{dy}{dx} = \frac{-x}{\sqrt{1 - x^2}}$

$f = \sqrt{u}$ $\quad u = 1 - x^2$

$f' = \frac{1}{2\sqrt{u}}$ $\quad u' = -2x$

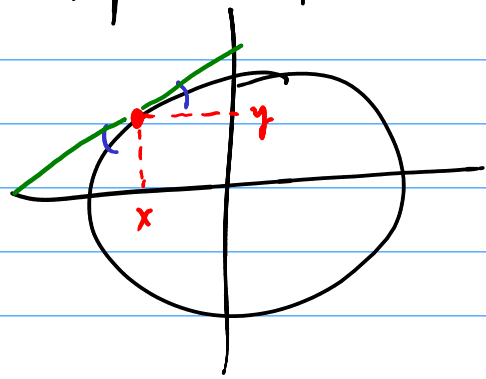
$$\left.\frac{dy}{dx}\right|_{x = 1/2} = \frac{-1/2}{\sqrt{1 - 1/4}} = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

tan. line: $y = \frac{\sqrt{3}}{2} + \frac{dy}{dx}\left(x - \tfrac{1}{2}\right) = \frac{\sqrt{3}}{2} + \left(\frac{-\sqrt{3}}{3}\right)\left(x - \tfrac{1}{2}\right)$

$$= -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{2}$$

$$\boxed{y = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}}$$

Given a curve, if we can cut out a small piece of the curve around a given point that passes the VLT, then we call this an implicit function: $y = y(x)$.



Need both x and y-coord's.

Ex. $x^2 + y^2 = 1$ if $y \neq 0$, then $y = y(x)$.

$$\frac{d}{dx}\left[x^2 + (y(x))^2 = 1\right]$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \qquad \text{Solve for } dy/dx.$$

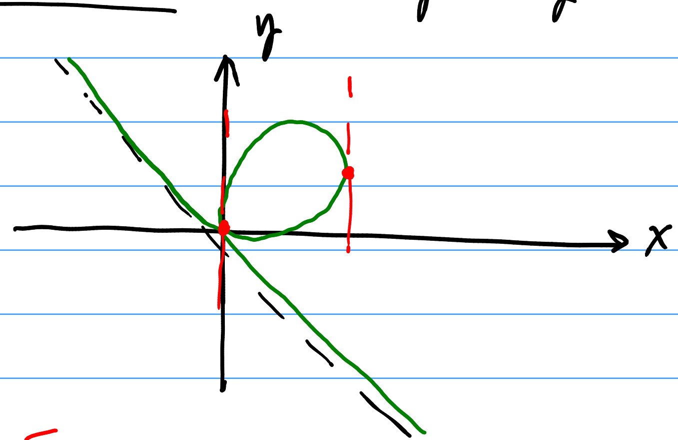
$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$ works for any pt on $x^2 + y^2 = 1$

Before: Top half: $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$ $= y$ on top half.

Ex. Folium of Descartes : $x^3+y^3=6xy$



Find $\frac{dy}{dx}$: $\frac{d}{dx}\left[x^3+y^3=6xy\right]$

$$3x^2+3y^2\cdot\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y-3x^2$$

$$\frac{dy}{dx}\left(3y^2-6x\right)=6y-3x^2$$

$$\boxed{\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}}=\frac{2y-x^2}{y^2-2x}$$

The folium vertical tangents when : $y^2-2x=0$

$$y^2=2x \iff \boxed{x=\frac{y^2}{2}}$$

$$x^3+y^3=6xy$$

$$\left(\frac{y^2}{2}\right)^3+y^3=6\left(\frac{y^2}{2}\right)y$$

$$\frac{y^6}{8}+y^3=3y^3 \to \frac{1}{8}y^6-2y^3=0$$

$$y^3\left(\frac{1}{8}y^3-2\right)=0$$

$$y^3 = 0 \longrightarrow y = 0.$$

$$\frac{1}{8}y^3 - 2 = 0$$

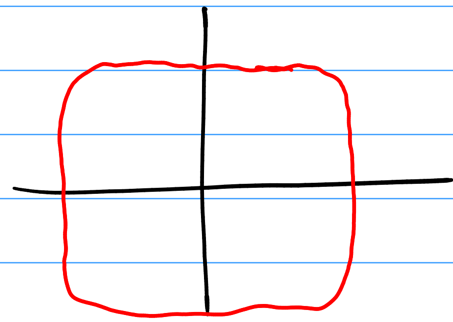
$$y^3 = 16$$

$$y = \sqrt[3]{16} = 2\sqrt[3]{2}$$

Ex. $x^k + y^k = r^k$ for k an even integer

Apple: $x^{16} + y^{16} = 1$ "Fat Circles"

Find dy/dx.



$$\frac{d}{dx}\left[x^{16} + y^{16} = 1\right]$$

$$16x^{15} + 16y^{15} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-16x^{15}}{16y^{15}} = \frac{-x^{15}}{y^{15}} = -\left(\frac{x}{y}\right)^{15}$$

Ex. Inverse Trig Functions.

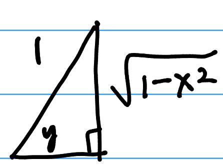
$y = \arccos(x)$ means $\cos(y) = x \leftarrow$ $0 \le y \le \pi$

$y = \cos^{-1}(x)$

Take the derivative (implicitly):

$$\frac{d}{dx}\left[\cos(y) = x\right]$$

$$\cos(y) = \frac{x}{1}$$



$$-\sin(y) \cdot \frac{dy}{dx} = 1$$

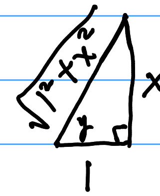
$$\frac{dy}{dx} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-x^2}}$$

$$1^2 = x^2 + b^2 \Rightarrow b = \sqrt{1-x^2}$$

$$\left(\cos^{-1}(x)\right)' = \frac{-1}{\sqrt{1-x^2}}$$

Ex. $\frac{d}{dx}\left[\arctan(x)\right] = ?$

$y = \arctan(x)$ __means__ $x = \tan(y)$



$$\frac{d}{dx}\left[x = \tan(y)\right]$$

$$1 = \sec^2(y)\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \left(\cos(y)\right)^2 = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

$$\left(\arctan(x)\right)' = \frac{1}{1+x^2}$$