M242 9/13/18

§2.8 – Related Rates

Idea: suppose two or more quantities both depend on the same independent variable and are related by an eq'n.

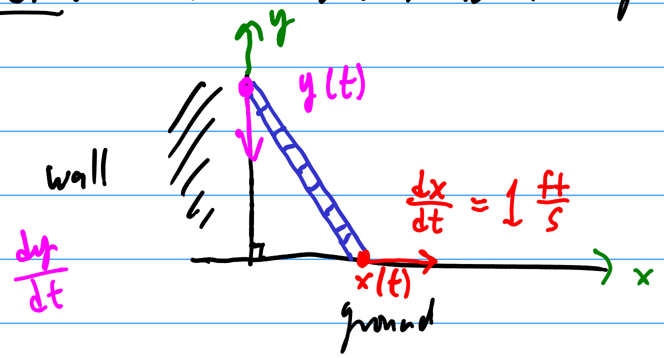
e.g., $A(t) + B(t) = k$

then their rates of change (i.e., derivatives) are also related.

$$\frac{d}{dt}\left[A(t) + B(t) = k\right]$$

$$A'(t) + B'(t) = 0$$

Ex. A 10 ft ladder is leaning on a wall.



The base of the ladder is sliding away from the wall at 1 ft/s.

How fast is the ladder sliding down the wall when the base is 6 ft from the wall?

TBD: $\frac{dy}{dt} = ?$

Known:

ladder $= c = 10$ ft, $x = 6$ ft $\Rightarrow$ $\boxed{y^2 + 6^2 = 10^2}$

$y = \sqrt{100-36} = \sqrt{64}$

$y = 8$ ft

$\frac{dx}{dt} = 1$ ft/s

Pyth. Thm: $x^2 + y^2 = 10^2$

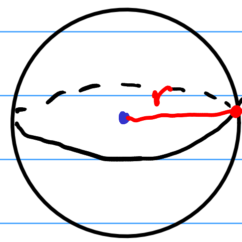
Take $\frac{d}{dt}$ of ↑

$$\frac{d}{dt}\left(x^2 + y^2 = 10^2\right) \Rightarrow 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$2(6)\cdot 1 + 2(8)\frac{dy}{dt} = 0$$

$$2(8)\frac{dy}{dt} = -2(6) \rightarrow \frac{dy}{dt} = -\frac{6}{8} = \frac{-3}{4} \text{ ft/s}$$

Ex. Air is being into a spherical balloon so that its volume is increasing at a rate of 100 cm^3/s.
How fast is the radius of the balloon increasing when its diameter is 50 cm?



$$\begin{cases} r = r(t) \\ V = \frac{4}{3}\pi r^3 \end{cases} \qquad V = V(t)$$

Known: $r = 25$ cm

$\frac{dV}{dt} = 100 \ cm^3/s$

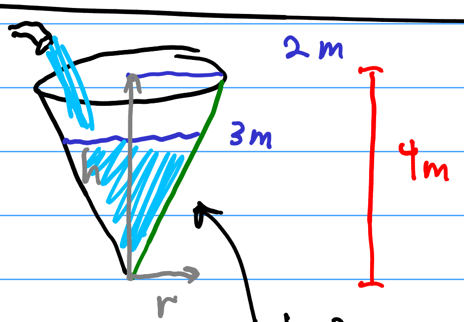
TBD: $\frac{dr}{dt}$

$$\frac{d}{dt}\left[V = \frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\cancel{100}}{\cancel{4}\pi (25)^{\cancel{2}}} = \boxed{\frac{1}{25\pi} \ \frac{cm}{s} = \frac{dr}{dt}}$$

Ex.



$h = 2r$

"Water Tank"

Water is being pumped into the tank at a rate of 2 m^3/min
At what rate is the water level rising when the height is 3m?

$$\frac{dV}{dt} = 2, \quad V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3$$

or

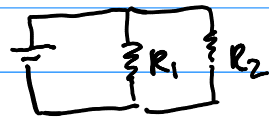
$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

$$\frac{d}{dt}\left[V = \frac{1}{12}\pi h^3\right]$$

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2\cdot \cancel{12}^{4}}{\cancel{3}\cdot 3^2 \cdot \pi} = \frac{8}{9\pi} \text{ m/min}$$

Ex. Ohm's Law: Two resistors w/ Resistances R_1 and R_2 are connected in parallel:



The total resistance R obeys,

$$\underbrace{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}} \quad \text{or} \quad R = \frac{R_1 R_2}{R_1 + R_2}$$

Known: $\frac{dR_1}{dt} = 0.3\ \Omega/s \qquad \frac{dR_2}{dt} = 0.2\ \Omega/s$

Find $\boxed{\frac{dR}{dt}}$ when $R_1 = 80\Omega \quad R_2 = 100\Omega$ ← $R = \frac{80 \cdot 100}{80 + 100} = \frac{800\cancel{0}}{18\cancel{0}} = \frac{400}{9}$

$$\frac{d}{dt}\left[\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$R^{-2} \cdot \frac{dR}{dt} = R_1^{-2}\frac{dR_1}{dt} + R_2^{-2}\frac{dR_2}{dt}$$

$$\frac{dR}{dt} = R^2\left[\frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}\right] = \left(\frac{400}{9}\right)^2\left(\frac{1}{80^2}\cdot\frac{3}{10} + \frac{1}{100^2}\cdot\frac{2}{10}\right)$$

$$\boxed{\frac{dR}{dt} \approx 0.132\ \Omega/s}$$ (check me)