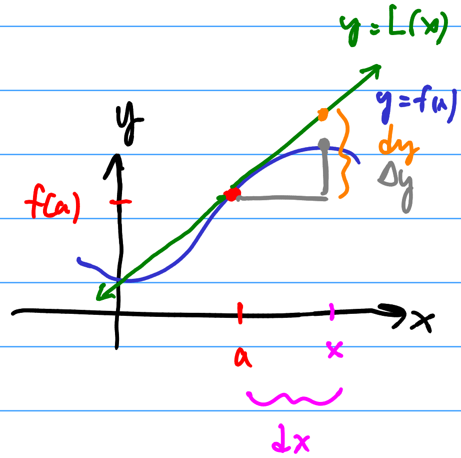
M242 9/17/18

§2.9 – Linear Approximations and Differentials

$y=f(x)$ differentiable at $x=a$



Linear Approximation: $\underset{L(x)}{\underbrace{y}} = f(a) + f'(a)(x-a)$

Differentials:

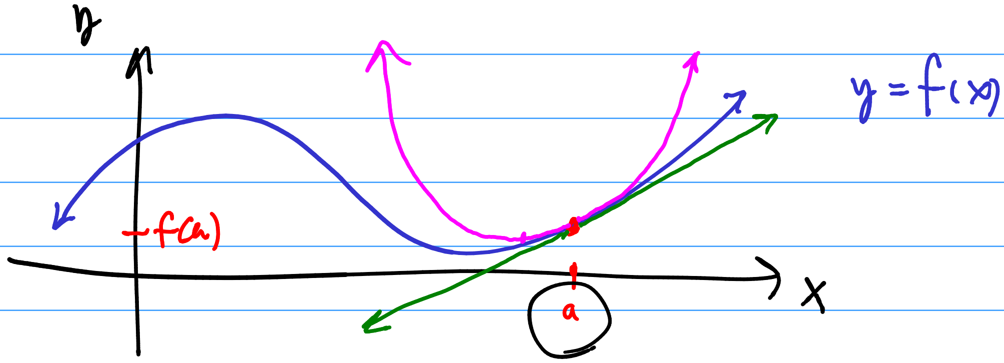
$dx = \Delta x$ = change in x $(\neq 0)$

Δy = change in y-value of the function

dy = change in y-value of the tangent line. $\quad dy = f'(a)\,dx$

$$f'(a) = \frac{dy}{dx}$$

Idea: $y=f(x)$ $\quad$ f has as many derivatives as we want to take.



Q. Find the best fit parabola to $y=f(x)$ at $x=a$:

Criteria:
1. $(a, f(a))$ is on the parabola
2. the first derivatives should coincide at $x=a$.
3. the second derivatives coincide at $x=a$.

General Eq'n of parabola: $p(x) = ax^2 + bx + c \qquad a \neq 0.$

OR

$$\rightarrow \quad p(x) = A(x-a)^2 + B(x-a) + C$$

$$p(a) = f(a) \leftarrow$$

1. $p(a) = A(\cancel{a-a})^2 + B(\cancel{a-a}) + C = C = f(a)$

$$p(x) = A(x-a)^2 + B(x-a) + f(a)$$

2. $p'(a) = f'(a) \leftarrow$

$$p'(x) = \left(A(x-a)^2 + B(x-a) + f(a)\right)'$$

$$\rightarrow \quad p'(x) = 2A(x-a) + B$$

$$p'(a) = 2A(\underbrace{\cancel{a-a}}_{0}) + B = B = f'(a)$$

$$p(x) = A(x-a)^2 + f'(a)(x-a) + f(a)$$

3. $p''(a) = f''(a)$

$$p''(x) = (p'(x))' = \left(2A(x-a) + B\right)'$$

$$= 2A = f''(a) \quad \Rightarrow \quad A = \frac{1}{2} f''(a)$$

$$\boxed{p(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2} \quad (*)$$

Def'n. Let f be an infinitely differentiable function at $x=a$. The n^{th} degree Taylor polynomial for f at a is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

x^3

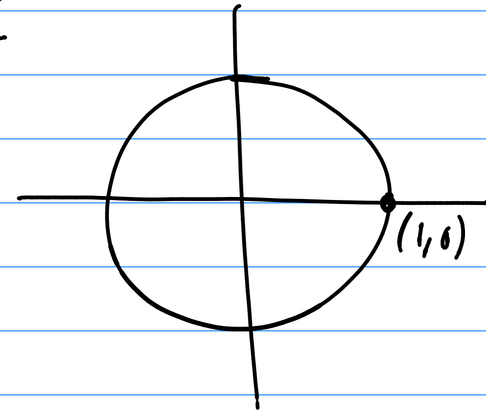
$3x^2$

$6x$

6

(*) $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Ex. $f(x) = \sin x$ Find $T_5(x)$ $\underline{\underline{a=0}}$



k	$f^{(k)}(x)$	$\xrightarrow{x=a}$	$f^{(k)}(0)$
0	$\sin x$		0
1	$\cos x$		1
2	$-\sin x$		$+0$
3	$-\cos x$		-1
4	$\sin x$		0
5	$\cos x$		1

$$T_5(x) = \sum_{k=0}^{5} \frac{f^{(k)}(0)}{k!}(x-a)^k$$

$$= \underset{0}{\cancel{f(0)}} + \underset{1}{f'(0)}\,x + \underset{0}{\cancel{\tfrac{1}{2} f''(0)\,x^2}} + \tfrac{1}{6}\underset{-1}{f'''(0)}\,x^3 + \underset{0}{\cancel{\tfrac{1}{24} f^{(4)}(0)\,x^4}} + \tfrac{1}{120}\underset{1}{f^{(5)}(0)}\,x^5$$

$$T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$