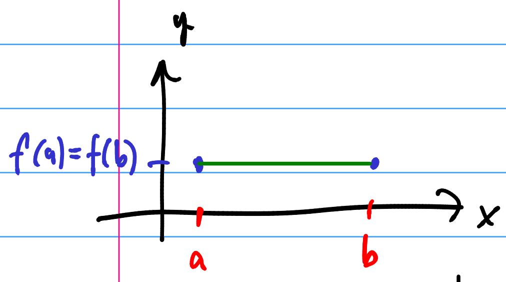
M242 9/24/18

§3.2 – Mean Value Theorem (MVT)

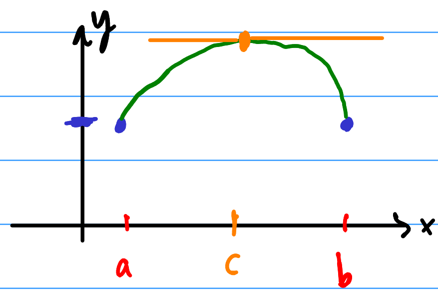
Thm. (Rolle's Theorem)
Let f be a continuous function on $[a,b]$, and differentiable on (a,b). Suppose $f(a)=f(b)$. Then there exists a number c, $a<c<b$, such that

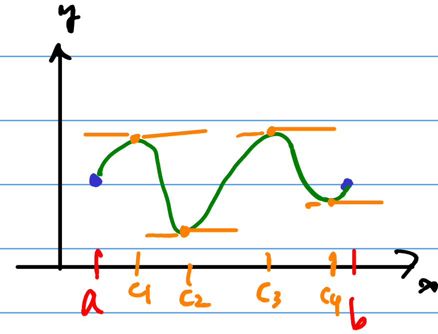
$$f'(c)=0.$$



Case I. f is constant

Every pt. satisfies the theorem.





"Proof:" Case I. If the function is constant on (a,b), then $f'(c)=0$ for every point in (a,b).

Case II. If f is not constant, then f' is continuous.
Then apply the IVT to f' to get the result. ∎

Ex. Show that $x^3+x-1=0$ has exactly one real root.

1. Show that there is at least one real root: IVT.

$f(x)=x^3+x-1$

$f(0)=0+0-1=-1<0$
$f(1)=1^3+1-1=1>0$
} by IVT there is a c s.t. $f(c)=0$, $0<c<1$.

2. Assume that f has 2 real zeros: a, b.

$$\underbrace{f(a) = f(b)}\approx 0$$

$f(x) = x^3 + x - 1$ is both continuous and differentiable, so Rolle's Theorem applies.

$$f'(c) = 0$$

but $f'(x) = 3x^2 + 1 \approx 0$

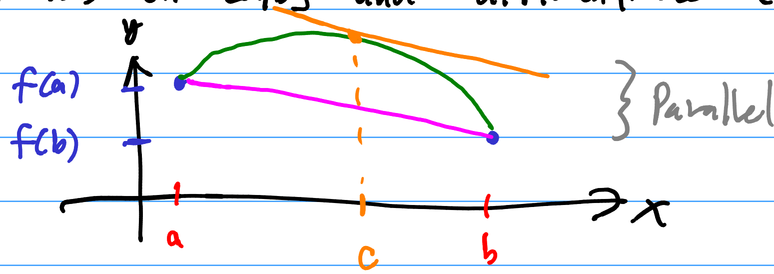
$x = \pm i\sqrt{1/3}$ NOT REAL

OR : $3x^2 + 1 \geq 1$ so it cannot $= 0$

This contradicts Rolle's Theorem! So our assumption that f has 2 real roots was false. ∎

Thm. (Mean Value Theorem ~ MVT)

Let f be continuous on [a,b] and differentiable (a,b).



Then there exists a number c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

"Proof": Secant line: $y = y_0 + m(x - x_0) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$

Make a new function $h(x) = f(x) -$ sec. line

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

This h satisfies the criteria of Rolle's Thm: cont., diff.,

$h(a) = 0 = h(b)$

Now Rolle's Thm says that there is a pt c such that

$$h'(c) = 0$$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

So,

$$h'(c) = \underline{f'(c) - \frac{f(b)-f(a)}{b-a} = 0}$$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\text{OR} \quad f'(c)(b-a) = f(b)-f(a) \qquad \blacksquare$$

Ex. $f(x) = x^3 - x$ $\quad$ $\underline{a=0, \ b=2}$ $\quad$ Find all c that satisfy the MVT.

1. $f(x) = x^3 - x$ is cont. and diff. on $\mathbb{R}$. So MVT applies.

2. $\frac{f(2)-f(0)}{2-0} = \frac{6-0}{2-0} = 3$

3. Solve $f'(x) = 3$ for x. Only keep sol'ns in $(0,2)$.

$$f'(x) = 3x^2 - 1 = 3$$
$$3x^2 = 4$$
$$x^2 = 4/3$$
$$x = \pm \frac{2}{\sqrt{3}}$$
$$\text{So, } \boxed{c = 2/\sqrt{3}}$$

Ex. f satisfies the hypotheses of MVT on $[0,2]$. And $f(0) = -3$ and $f'(x) \le 5$ on $[0,2]$.

Q. what is the max. value that $f(2)$ can attain?

MVT: $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\quad a=0,\ b=2$

$$f'(c) = \frac{f(2)-f(0)}{2-0} = \underbrace{\frac{f(2)+3}{2}} = \underbrace{f'(c) \leq 5}$$

Then $f(2) \leq 2 \cdot 5 - 3 = 7$