M242 9/28/18

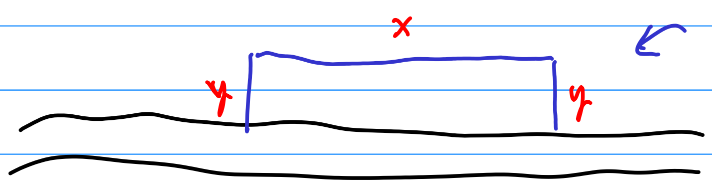
§3.7 – Optimization

f has critical pts at $x=a$ if $f'(a) = \begin{cases} 0 \\ \text{undef.} \end{cases}$

and a ∃ in the domain.

Fermat's Thm. If $(a, f(a))$ is a local extrema, then $(a, f(a))$ is a critical pt.

Ex.



construct a fence w/ 2400 ft of fence

construct the pen w/ largest area given this amount of fence.

Constraint: $P = \boxed{x + 2y = 2400}$

T.B.O.: $A = xy$ Find the maximum value of A.

Solve for x: $x = 2400 - 2y$

then, $A(y) = (2400 - 2y)y$ Fermat's Thm applies! ☺

$$A(y) = 2400y - 2y^2$$

$$\rightarrow A'(y) = \underbrace{2400 - 4y}_{} = 0$$

$$4y = 2400$$

$$y = 600 \text{ ft.}$$

$$x = 2400 - 2(600)$$

$$= 2400 - 1200$$

$$x = 1200 \text{ ft.}$$

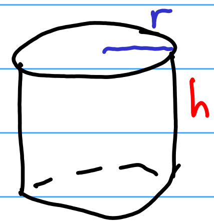
Does (1200, 600) correspond to a max?

$A''(y) = -4 < 0$ Concave down.

The pt we found is indeed a maximum!

$A(1200, 600) = 1200 \cdot 600 = 720{,}000 \text{ ft}^2$

Ex.

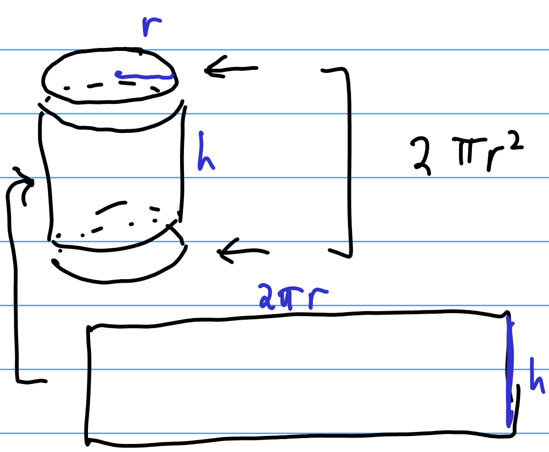


Find the (r,h) that minimize surface area w/ volume = 1L.

Constraint: $V = 1L = \pi r^2 h = 1L = 1000 \text{ mL} = 1000 \text{ cm}^3$

$h, r > 0$

h, r are measured in cm



$SA = 2\pi r^2 + 2\pi r h$

"To be optimized"

recall $\pi r^2 h = 1000 \leftarrow$ replace h

$SA(r) = 2\pi r^2 + 2\cancel{\pi r}\left(\frac{1000}{\cancel{\pi} r^{\cancel{2}}}\right)$

$SA(r) = 2\pi r^2 + \frac{2000}{r}$

$SA'(r) = 4\pi r - \frac{2000}{r^2} = 0$

$4\pi r = \frac{2000}{r^2}$

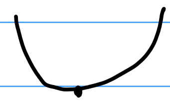
$4\pi r^3 = 2000$ $\qquad r = \sqrt[3]{\frac{2000}{4\pi}} = 10\sqrt[3]{\frac{2}{4\pi}}$

$\boxed{r = \frac{10}{\sqrt[3]{2\pi}}}$

Is it a max/min?

$$SA'(r) = 4\pi r - \frac{2000}{r^2}$$

$$SA''(r) = 4\pi + \frac{4000}{r^3} \qquad w/ \ r>0$$



so $SA''(r) > 0$

So $\boxed{r = \frac{10}{\sqrt[3]{2\pi}}}$ corresponds to a min.

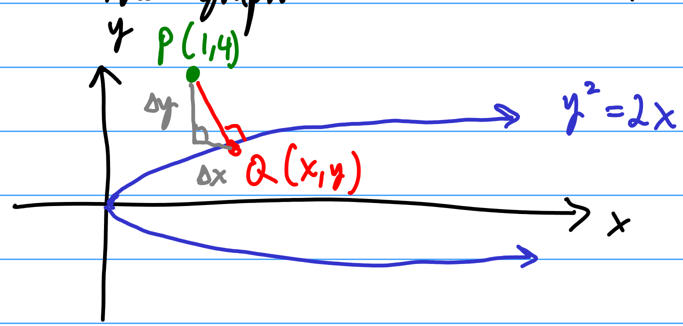
$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\frac{10}{\sqrt[3]{2\pi}}\right)^2} = \frac{\cancel{1000}\,(2\pi)^{2/3}}{\cancel{100}\,\pi} = \boxed{\frac{10\sqrt[3]{4}}{\sqrt[3]{\pi}} = h}$$

both in cm.

Ex. $y^2 = 2x$ $\qquad P(1,4)$

$4^2 \stackrel{?}{=} 2(1) \rightarrow 16 \neq 2$ $\qquad$ so P is not on the graph.

Find the point on the graph that is closest to $P(1,4)$.



$x = \frac{1}{2}y^2$

Find (x,y) s.t. the distance is minimum.

$$d(P,Q) = \sqrt{\Delta x^2 + \Delta y^2}$$

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$$d^2(y) = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2}^{\,2}$$

T.B.C.

Because distance is non-negative and $\sqrt{\ }$ is increasing, we can optimize

$$f = d^2 = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

$$f' = 2\left(\tfrac{1}{2}y^2 - 1\right)\cdot \cancel{\tfrac{1}{2}}\cdot\cancel{2}y + 2(y-4) = 0$$

$$= y^3 \underbrace{-2y + 2y}_{} - 8 = 0$$

$$y^3 = 8$$

$$\boxed{y = 2} \qquad x = \tfrac{1}{2}(2)^2 = 2$$

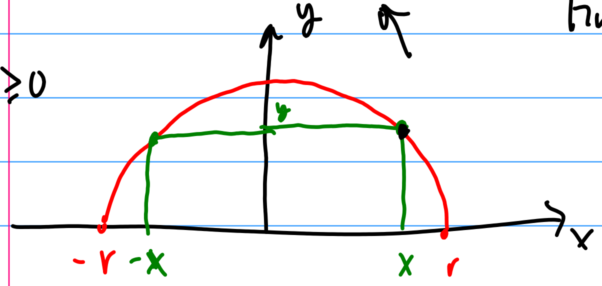
So, the pt. $Q(2,2)$.

<u>RE. Check that it's a min</u>

$$f'(y) = y^3 - 8 \qquad f''(y) = 3y^2 \geq 0$$

<u>Ex</u> $x^2 + y^2 = r^2$

$y \geq 0$



Find the largest (area) rectangle that can be inscribed this way.

<u>TBO</u>: $A = 2xy$

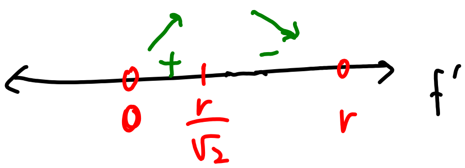
$y = \sqrt{r^2 - x^2}$ constraint

$A(x) = 2x\sqrt{r^2 - x^2}$ <u>optimize!</u>

$$A'(x) = 2\sqrt{r^2 - x^2} + 2\cancel{x}\frac{-\cancel{2}x^2}{\cancel{2}\sqrt{r^2 - x^2}} = 2\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}} = 0 \quad \text{only if} \quad r^2 = 2x^2$$

so $x = \frac{r}{\sqrt{2}}$

This is indeed a max by the FDT:



Now $y = \sqrt{r^2 - x^2} = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}}$.

So the maximum area of the inscribed area is

$$A_{max} = 2\left(\frac{r}{\sqrt{2}}\right)\left(\frac{r}{\sqrt{2}}\right) = 2\,\frac{r^2}{2} = r^2.$$