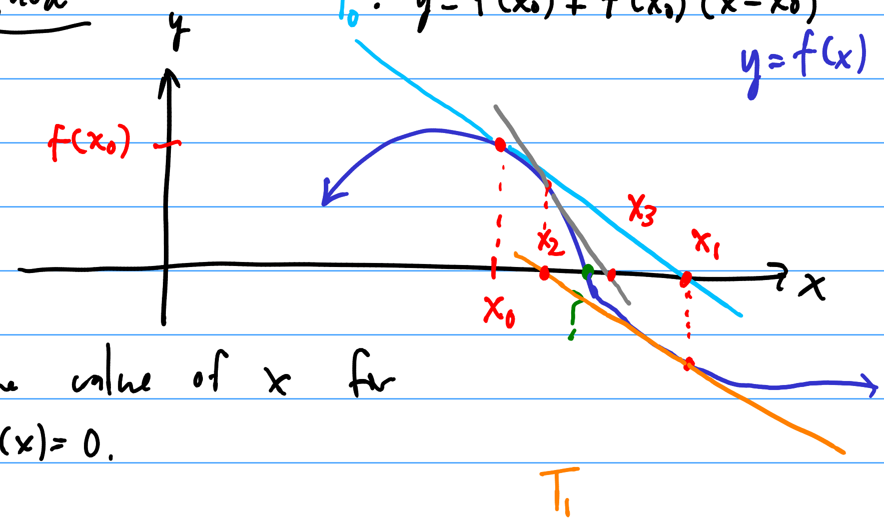
M242 10/2/18

§3.8 – Newton's Method

$y = f(x)$ smooth

$T_0: \; y = f(x_0) + f'(x_0)(x - x_0)$



Goal: determine the value of x for which $f(x) = 0$.

$\cos x = x$ $\leftarrow$ $\underbrace{\cos x - x}$ $= 0$

$x = \cos^{-1}(x)$

$x = 0$: $1 > 0$

$x = \pi$: $-1 - \pi < 0$

Algorithm: 1. Start w/ a smooth function f, and consider $f(x) = 0$.

e.g., $\cos(x) = x \Rightarrow \underbrace{\cos(x) - x}_{f(x)} = 0$

2. Make an initial guess $x = x_0$.

3. update the x-value:
the equation of the tangent line at step k is:

$$y = f(x_k) + f'(x_k)(x - x_k)$$

Find x-int: $0 = f(x_k) + f'(x_k)(x - x_k)$ Solve for x.

$$-f(x_k) = f'(x_k)(x - x_k)$$

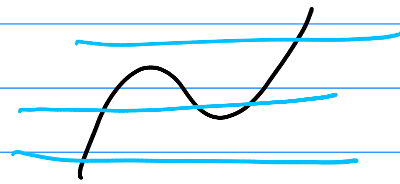
$$\frac{-f(x_k)}{f'(x_k)} = x - x_k$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \quad (*)$$

Newton's Method Formula.

4. Repeat until we achieve a desired tolerance.

<u>Ex.</u> $\begin{cases} x^3 - 2x - 5 = 0 \\ x_0 = 2 \end{cases}$



Find x_2 from Newton's method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2$$

$$x_{k+1} = x_k - \frac{x_k^3 - 2x_k - 5}{3x_k^2 - 2}$$

$x_0 = 2$

$$x_1 = 2 - \frac{2^3 - 2\cdot 2 - 5}{3\cdot 2^2 - 2} = 2 - \frac{-1}{10} = \frac{21}{10} = 2.1$$

$$x_2 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3\cdot(2.1)^2 - 2} \approx 2.0946$$

<u>Ex.</u> Estimate the value of $\sqrt[6]{2}$.

$$x^6 = \left(\sqrt[6]{2}\right)^6 \qquad \text{Find } x.$$

$$x^6 = 2$$

$$\underbrace{x^6 - 2}_{f(x)} = 0$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\boxed{x_{k+1} = x_k - \frac{x_k^6 - 2}{6x_k^5}} \quad (*)$$

$x_0 = 1$

get $x_5 = 1.2246\ldots$ correct to 16 decimals.

Ex. $\begin{cases} x^3 - 3x + 6 = 0 \\ x_0 = 1 \end{cases}$

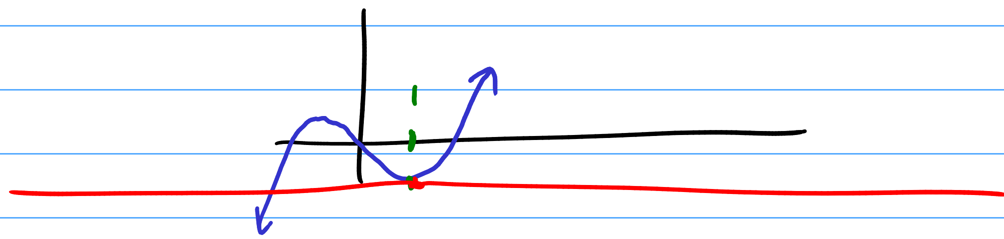
Newton's Method will not work for this guess. why?

$f(x) = x^3 - 3x + 6$ ← graph, then avoid peaks and valleys.

$f'(x) = 3x^2 - 3 = 0 \qquad 3(x+1)(x-1) = 0$

$x = 1, -1$ are CN's.

$x_1 = 1 - \frac{4}{0}$ ↖ Not allowed!



Horizontal Tan. Line