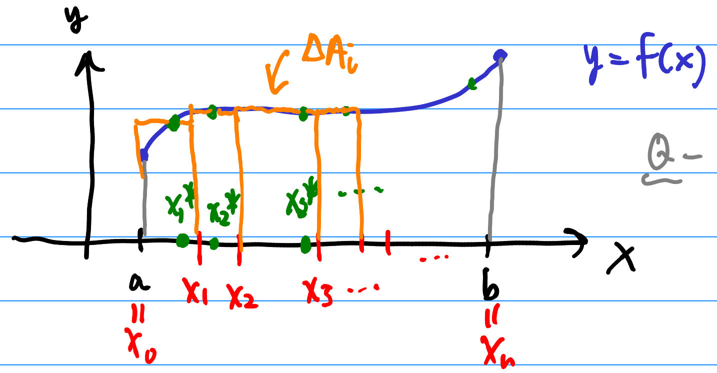
M242 10/12/18

Ch4 – Area under a curve

$y=f(x)$ continuous on $[a,b]$, $f(x)>0$ on $(a,b]$



Q – What is the area?

n is a positive integer = "number of boxes"

The $\{x_i\}$ form a partition of $[a,b]$. $a=x_0 < x_1 < x_2 < \cdots < x_n = b$

The lengths of each sub-interval are

$$\Delta x_i = x_i - x_{i-1} \qquad i=1,2,3,\ldots,n$$

Choose x_i^* in $[x_{i-1}, x_i]$, and then consider $(x_i^*, f(x_i^*))$

Area under curve is approximately $\sum_{i=1}^{n} \Delta A_i$

For each rectangle, $\Delta A_i = b \cdot h = f(x_i^*) \cdot \Delta x_i$

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

→ The exact area under the curve $y=f(x)$ over $[a,b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i \qquad \text{provided the limit exists.}$$

Preliminaries :

$$\boxed{\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n\text{-copies}} = n}$$

$$\sum_{i=1}^{n} i = \underbrace{1+2+3+4+\cdots+n}\ = S \qquad \text{so, } \boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$\begin{aligned} S &= 1+2+3+\cdots+n \\ + \quad S &= n+(n-1)+(n-2)+\cdots+1 \end{aligned}$$

$$\frac{2S}{2} = \underbrace{(n+1)+(n+1)+(n+1)+\cdots+(n+1)}_{n\text{-copies}} = \frac{n\cdot(n+1)}{2}$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}} \qquad \sum_{i=1}^{n} i^2 = 1+4+9+16+25+\cdots+n^2$$

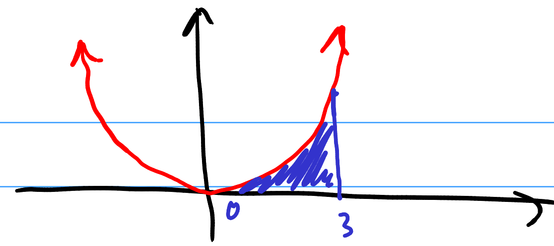
$$\boxed{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Conventions :

1. Set $\boxed{\Delta x_i = \frac{b-a}{n}}$ evenly spaced intervals
2. Use right endpoints for x_i^*
   Start w/
   $x_0 = a$
   $\boxed{x_i^* = a + i\Delta x}$

$\underline{\text{Ex}}$. $f(x) = x^2$ on $[0,3]$



$L_3 = 5$
$R_3 = 14$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \boxed{f(x_i^*)}\boxed{\Delta x_i}$$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \boxed{\frac{3}{n}}$$

$a = 0$

$$x_i^* = a + i\Delta x = 0 + i\frac{3}{n} = \frac{3}{n} i$$

$$f(x_i^*) = (x_i^*)^2 = \left(\frac{3}{n} i\right)^2 = \boxed{\frac{9}{n^2} i^2}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9}{n^2} i^2 \frac{3}{n} = \lim_{n\to\infty} \frac{27}{n^3} \boxed{\sum_{i=1}^{n} i^2}$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{27 \cdot (n^2+n)(2n+1)}{6n^3}$$

$$= \lim_{n\to\infty} \boxed{\frac{27}{6}} \cdot \underbrace{\frac{(2n^3+3n^2+n)}{n^3}}_{} \quad \frac{1/n^3}{1/n^3}$$

$$= \frac{27}{6} \lim_{n\to\infty} \frac{2 + 3/n + 1/n^2}{1} = \frac{27}{6} \cdot \frac{2}{1} = \frac{27}{3} = 9$$

Exact area is 9!

$\underline{\text{Notation}}$: $$A = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n}}_{} \underbrace{f(x_i)}_{}\underbrace{\Delta x}_{} = \int_a^b f(x)\, dx$$

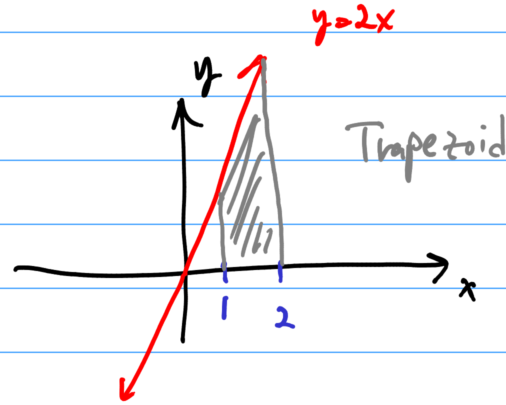
Riemann Sum

Definite Integral

$\underline{\text{Ex}}$. $$\int_0^3 x^2\, dx = 9$$

$\underline{Ex.}$ $\int_1^2 2x\,dx$ = Area under $y=2x$ on $[1,2]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$



$$A = \tfrac{1}{2}\left(f(1)+f(2)\right)\cdot(2-1)$$

$$= \tfrac{1}{2}(2+4)(1) = 3$$

$\underline{R.S.}$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$a=1$

$x_i = a + i\Delta x = 1 + i\frac{1}{n}$

$f(x_i) = 2\left(1+i\frac{1}{n}\right) = 2 + i\frac{2}{n}$

$$\int_1^2 2x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{2i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\left(\overbrace{\sum_{i=1}^{n}\frac{2}{n}}^{\frac{2}{n}\sum_{i=1}^{n}1} + \overbrace{\sum_{i=1}^{n}\frac{2i}{n^2}}^{\frac{2}{n^2}\sum_{i=1}^{n}i}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{\cancel{n}}\cdot\cancel{n} + \frac{\cancel{2}}{n^2}\,\frac{n(n+1)}{\cancel{2}}\right)$$

$$= \lim_{n\to\infty}\left(2 + \frac{1+\frac{1}{n}}{1}\right) = 2+1 = 3$$