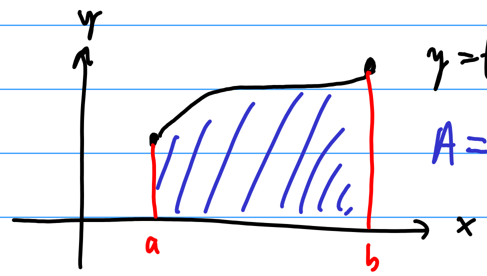
M242 23 Oct '18

Ch4 - Integration

Definite Integrals: $\int_a^b f(x)\,dx =$ "Area under $y=f(x)$ over $a \le x \le b$"



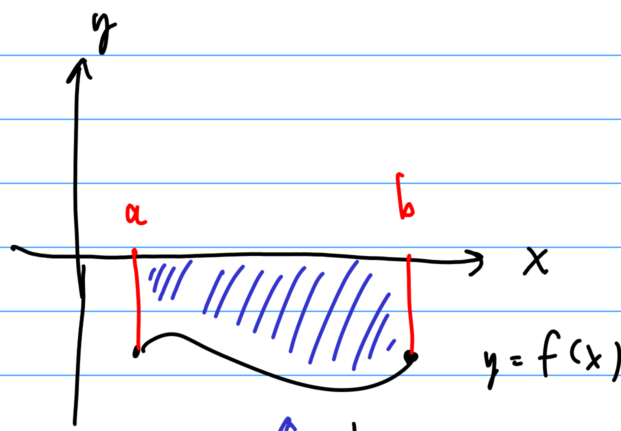
$A = \int_a^b f(x)\,dx$

$$\int_b^a f(x)\,dx = -\text{Area}$$

reversed direction

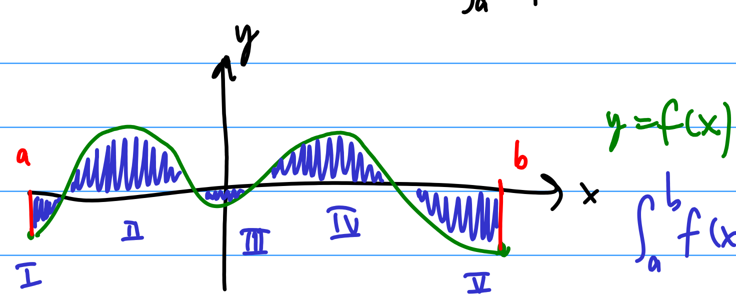
$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx} \quad (*)$$

Ex. $f(x) < 0$



$\int_a^b f(x)\,dx < 0$

Ex.



$\int_a^b f(x)\,dx = (\text{II} + \text{IV}) - (\text{I} + \text{III} + \text{V})$

Ex. $\int_{-2}^{2}\sqrt{4-x^2}\,dx = \int_{-2}^{2}(4-x^2)^{1/2}\,dx$

$\cancel{4x - \frac{x^3}{3}}$

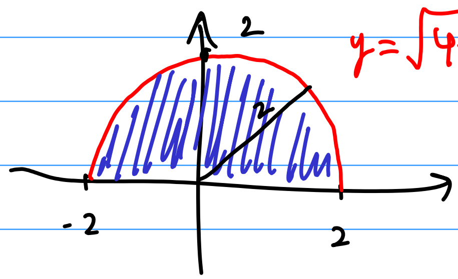
$\frac{-1}{x}(4-x^2)^{3/2} \xrightarrow{d/dx} \frac{3}{2}(4-x^2)^{1/2}(-2x)$

Geometry:

$y^2 = \sqrt{4-x^2}^{\,2}$

$y^2 = 4-x^2$

$x^2+y^2 = 2^2 \qquad y \geq 0$

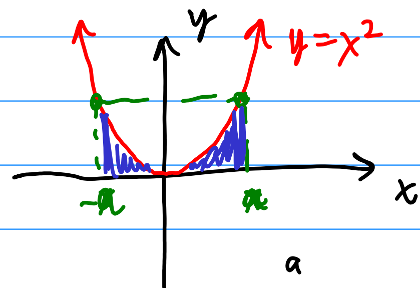


Area $= \pi r^2$, $r = 2$

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4) = 2\pi$$

Properties:

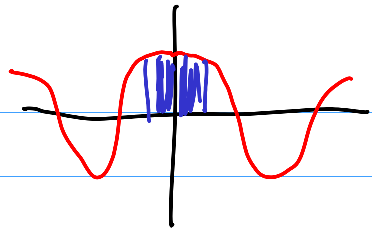
Even function: f is even if its graph has symmetry over y-axis.



$f(-x) = f(x)$

For even functions only: $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Ex. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos\theta \, d\theta$



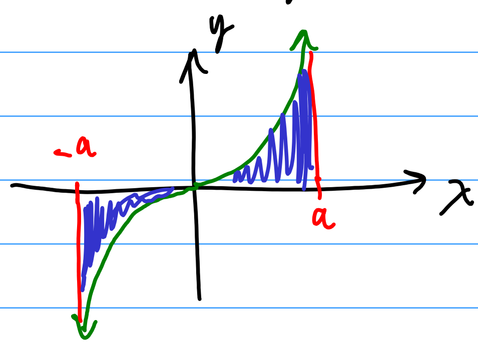
$$= 2\int_0^{\pi/4} \underbrace{\cos\theta}\, d\theta = 2\cdot\sin\theta \Big|_{\theta=0}^{\pi/4} = 2\sin(\pi/4) - 2\sin 0$$

$\uparrow$ FTC

$$= 2\frac{\sqrt{2}}{2} - 0$$

$$= \sqrt{2}.$$

Ex. A function f is odd if its graph has rotational symmetry or symmetry about the origin.



odd functions only: $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

Ex. $\displaystyle\int_{-1}^{1} x^3 - 3x^2 + 2x + 1 \, dx = \underbrace{\int_{-1}^{1} \cancel{x^3 + 2x}\, dx}_{\text{ODD}} \overset{0}{} + \underbrace{\int_{-1}^{1} -3x^2 + 1 \, dx}_{\text{EVEN}}$

$$= 2\int_0^1 -3x^2 + 1 \, dx = 2\left[-\frac{3}{3}x^3 + x\right]\Bigg|_{x=0}^{1}$$

$$= 2\left[-1^3 + 1 - (-0^3 + 0)\right] = 2(0) = \underline{0}.$$

Ex. Riemann Sum: $\int_1^2 \underbrace{2x^2+x}_{f(x)}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i^* = a + i\Delta x = 1 + i\frac{1}{n}$$

$$f(x_i^*) = 2\left(1+i\frac{1}{n}\right)^2 + \left(1+i\frac{1}{n}\right)$$

$$= 2\left(1 + \frac{2}{n}i + \frac{1}{n^2}i^2\right) + 1 + \frac{1}{n}i$$

$$= 3 + \frac{5}{n}i + \frac{2}{n^2}i^2$$

$$\int_1^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n} + \frac{5}{n^2}i + \frac{2}{n^3}i^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\underbrace{\sum_{i=1}^{n} 1} + \frac{5}{n^2}\underbrace{\sum_{i=1}^{n} i} + \frac{2}{n^3}\underbrace{\sum_{i=1}^{n} i^2}\right)$$

$$= \lim_{n\to\infty} \left(\underbrace{\frac{3}{n}\cdot n} + \underbrace{\frac{5}{n^2}\cdot\frac{n(n+1)}{2}} + \underbrace{\frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}}\right)$$

$$= 3 + \frac{5}{2} + \frac{2}{3} = \boxed{\frac{37}{6}}$$