M242 — 25 oct 18

Ch 4 Review

1. Compute $\int_0^2 x^2 - x \, dx$ using the Riemann Sum definition.
2. Compute $\int_2^4 x^2 \, dx$ using the Riemann Sum definition.
3. Evaluate the definite and indefinite integrals:
   a) $\int_0^1 (1-x^9)\, dx$
   b) $\int_0^1 (1-x)^9 \, dx$
   c) $\int_0^1 v^2 \cos(v^3)\, dv$
   d) $\int \frac{x+2}{\sqrt{x^2+4x}}\, dx$
   e) $\int_0^2 y^2 \sqrt{1+y^3}\, dy$
   f) $\int_{-1}^1 \frac{\sin x}{1+x^2}\, dx$
   g) $\int_0^3 |x^2-4|\, dx$
   h) $\int_{-9}^0 \sqrt{81-u^2}\, du$
4. Compute the derivatives of the functions.
   a) $g(x) = \int_0^{x^4} \cos(t^2)\, dt$
   b) $h(y) = \int_{\sqrt{y}}^{y} \frac{\cos\theta}{\theta}\, d\theta$

Brief Solutions:

1. $\int_0^2 x^2 - x \, dx$ $a=0,\ b=2,\ f(x)=x^2-x$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + i\frac{2}{n} = i\frac{2}{n}$

$f(x_i) = \left(i\frac{2}{n}\right)^2 - \left(i\frac{2}{n}\right) = \frac{4}{n^2}i^2 - \frac{2}{n}i$

$$\int_0^2 x^2 - x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n^2}i^2 - \frac{2}{n}i\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8}{n^3}i^2 - \frac{4}{n^2}i\right)$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{8}{n^3}i^2 - \sum_{i=1}^{\infty}\frac{4}{n^2}i\right)$$

$$= \lim_{n\to\infty}\left(\frac{8}{n^3}\sum_{i=1}^{n}i^2 - \frac{4}{n^2}\sum_{i=1}^{n}i\right)$$

$$= \lim_{n\to\infty}\left(\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\frac{n(n+1)}{2}\right)$$

$$= \frac{16}{6} - \frac{4}{2} = \frac{8}{3} - 2 = \frac{8}{3} - \frac{6}{3} = \boxed{\frac{2}{3}}$$

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2. $\int_2^4 x^2\,dx \qquad a=2,\ b=4,\ f(x)=x^2$

$$\Delta x = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$$
$$x_i = a + i\Delta x = 2 + i\frac{2}{n}$$
$$f(x_i) = 2^2\left(1+\frac{i}{n}\right)^2 = 4\left(1+\frac{2}{n}i+\frac{1}{n^2}i^2\right)$$

$$\int_2^4 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} 4\left(1+\frac{2}{n}i+\frac{1}{n^2}i^2\right)\left(\frac{2}{n}\right)$$
$$= 4\lim_{n\to\infty}\left(\frac{2}{n}\sum_{i=1}^{n}1+\frac{4}{n^2}\sum_{i=1}^{n}i+\frac{2}{n^3}\sum_{i=1}^{n}i^2\right)$$
$$= 4\lim_{n\to\infty}\left(\frac{2}{n}\cdot n+\frac{4}{n^2}\cdot\frac{n(n+1)}{2}+\frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$
$$= 4\left(2+\frac{4}{2}+\frac{4}{6}\right) = 4\left(\frac{12+2}{3}\right) = \frac{4\cdot 14}{3} = \boxed{\frac{56}{3}}$$

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3. a) $\int_0^1 (1-x^9)\,dx = x-\frac{1}{10}x^{10}\Big|_0^1 = 1-\frac{1}{10}1^{10}-\left(0-\frac{1}{10}0^{10}\right) = 1-\frac{1}{10} = \boxed{\frac{9}{10}}$

b.) $-\int_0^1 (1-x)^9\,-dx = -\int_1^0 u^9\,du = \int_0^1 u^9\,du = \frac{1}{10}u^{10}\Big|_0^1 = \frac{1}{10}-0 = \boxed{\frac{1}{10}}$

$u = 1-x \qquad u(1) = 1-1 = 0$

$du = -1\,dx \qquad u(0) = 1-0 = 1$

c.) $\frac{1}{3}\int_0^1 3v^2\cos(v^3)\,dv = \frac{1}{3}\int_0^1 \cos(u)\,du = \frac{1}{3}\sin(u)\Big|_0^1 = \frac{1}{3}\sin(1)-\frac{1}{3}\sin(0) = \boxed{\frac{1}{3}\sin(1)}$

$u = v^3 \qquad u(1) = 1^3 = 1$

$du = 3v^2\,dv \qquad u(0) = 0^3 = 0$

d.) $\frac{1}{2}\int \frac{(x+2)\cdot 2}{\sqrt{x^2+4x}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2}+C = \boxed{\sqrt{x^2+4x}+C}$

$u = x^2+4x$

$du = (2x+4)\,dx = 2(x+2)\,dx$

e.) $\frac{1}{3}\int_0^2 3y^2\sqrt{1+y^3}\,dy = \frac{1}{3}\int_1^9\sqrt{u}\,du = \frac{1}{3}\frac{2}{3}u^{3/2}\Big|_1^9 = \frac{2}{9}\left(\sqrt{9}^3-\sqrt{1}^3\right) = \frac{2}{9}(27-1) = \boxed{\frac{52}{9}}$

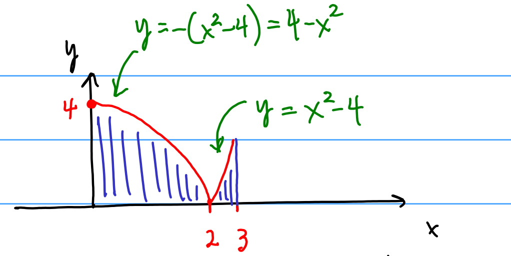
$u = 1+y^3 \qquad u(2) = 1+2^3 = 1+8 = 9$

$du = 3y^2\,dy \qquad u(0) = 1+0^3 = 1$

f.) $\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx$

$f(x) = \frac{\sin(x)}{1+x^2} \Longrightarrow f(-x) = \frac{\sin(-x)}{1+(-x)^2} = \frac{-\sin x}{1+x^2} = -f(x)$! so f is odd. Since the interval is symmetric and f is odd, then $\int_{-1}^1 f(x)\,dx = 0$.

g.) $\int_0^3 |x^2-4|\,dx$ Consider the graph:



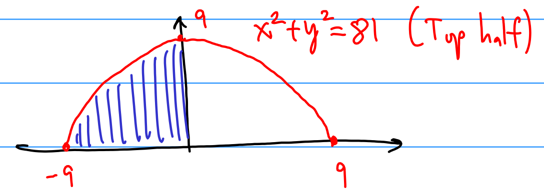
$$= \int_0^2 4-x^2\,dx + \int_2^3 x^2-4\,dx$$

Must be treated as 2 separate regions.

$$= \left(4x-\tfrac{1}{3}x^3\right)\Big|_0^2 + \left(\tfrac{1}{3}x^3-4x\right)\Big|_2^3 = \left(8-\tfrac{8}{3}\right)-\left(0-\tfrac{0}{3}\right)+\left(9-12\right)-\left(\tfrac{8}{3}-8\right)$$

$$=(8+8+9-12)-\tfrac{8}{3}-\tfrac{8}{3} = 13-\tfrac{16}{3} = \tfrac{39-16}{3} = \boxed{\tfrac{23}{3}}$$

h.) $\int_{-9}^{0} \sqrt{81-x^2}\,dx$ Consider the graph:



$$= \tfrac{1}{4}\text{ Area of disk w/ radius } 9 = \tfrac{1}{4}\pi(9)^2 = \boxed{\tfrac{81\pi}{4}}$$

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4. a.) $g(x)=\int_0^{x^4}\cos(t^2)\,dt$ Fundamental theorem of calculus!

if $f(x)=\int_0^x \cos(t^2)\,dt$, and $u(x)=x^4$, then $g(x)=f(u)$. Chain Rule!

$$\frac{d}{dx}\left[\int_0^{x^4}\cos(t^2)\,dt\right] = \cos(u^2)\cdot\frac{du}{dx} = \boxed{\cos(x^8)\cdot 4x^3}$$

b.) Same scenario! (kind of)

$$h(y)=\int_{\sqrt{y}}^{y}\frac{\cos\theta}{\theta}\,d\theta = \int_{\sqrt{y}}^{0}\frac{\cos\theta}{\theta}\,d\theta + \int_0^y\frac{\cos\theta}{\theta}\,d\theta = \int_0^y\frac{\cos\theta}{\theta}\,d\theta - \int_0^{\sqrt{y}}\frac{\cos\theta}{\theta}\,d\theta$$

$$\frac{d}{dy}\left[h(y)\right] = \frac{2\cos(y)}{2y} - \frac{\cos(\sqrt{y})}{\sqrt{y}}\cdot\frac{1}{2\sqrt{y}} = \boxed{\frac{2\cos y-\cos\sqrt{y}}{2y}}$$