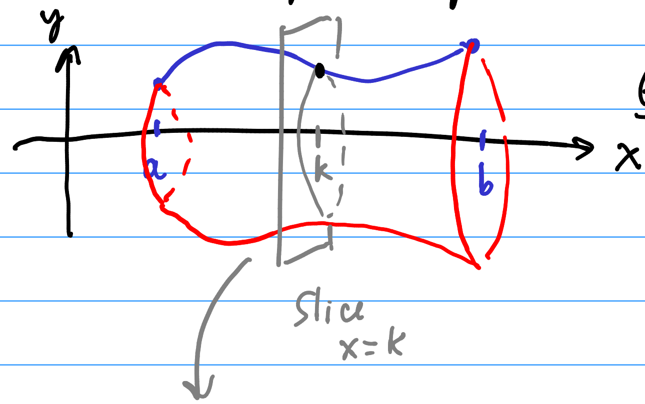
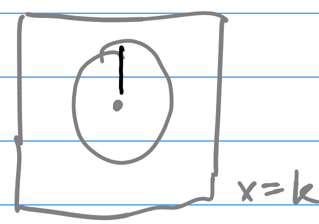
§5.2 - Volumes by slicing method

For now, let $y=f(x)$ be a positive function on $[a,b]$.

Create a 3D region by rotating the curve around the x-axis



Goal: Compute the volume



$A = \pi r^2 = \pi (f(x))^2$

So, the volume is $\boxed{V = \int_a^b A(x)\,dx} = \boxed{\pi \int_a^b (f(x))^2\,dx}$

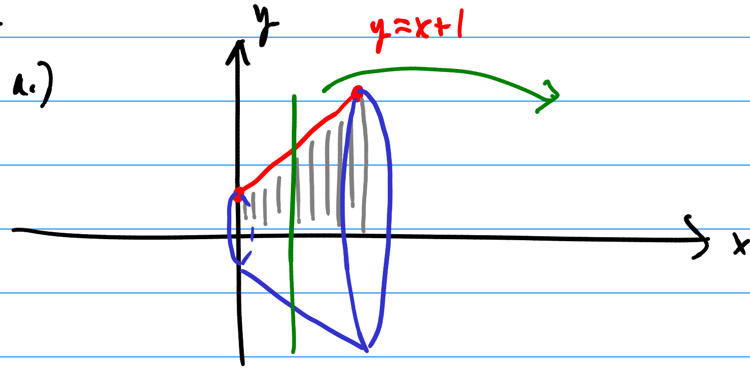
Ex. $\begin{cases} y = x+1 \\ y=0 \\ x=0 \\ x=2 \end{cases}$ region to be rotated is bdd between these curves.

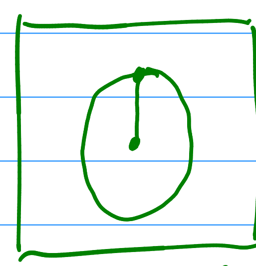
Then rotate about:

a.) x-axis

b.) $y=3$

a.)



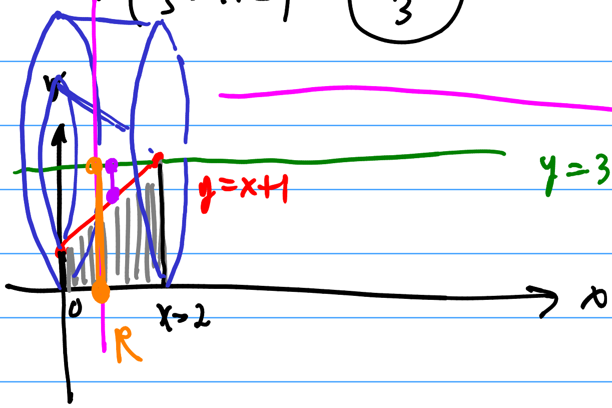


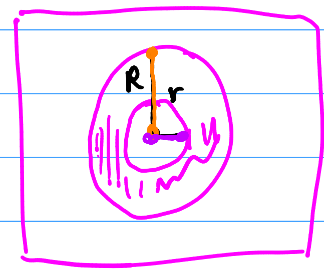
$A = \pi r^2$
$= \pi (x+1)^2$

$$V = \pi\int_0^2 (x+1)^2\,dx = \pi\int_0^2 x^2+2x+1\,dx = \pi\left(\tfrac{1}{3}x^3+x^2+x\right)\Big|_0^2$$

$$= \pi\left(\tfrac{8}{3}+4+2\right) = \frac{26\pi}{3}$$

b.)





$R=3$

$r = 3-(x+1)$

So, $A(x) = \pi R^2 - \pi r^2$

$= 9\pi - \pi(3-(x+1))^2$

$$V = \pi\int_0^2 9-(2-x)^2\,dx = \pi\int_0^2 9\,dx - \pi\int_0^2 x^2-4x+4\,dx$$

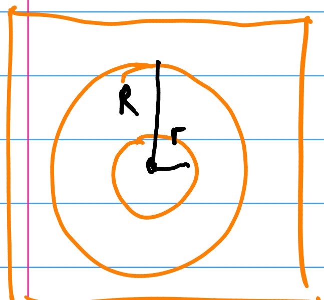
$$= \pi\, 9x\Big|_0^2 - \pi\left(\tfrac{1}{3}x^3-2x^2+4x\right)\Big|_0^2$$

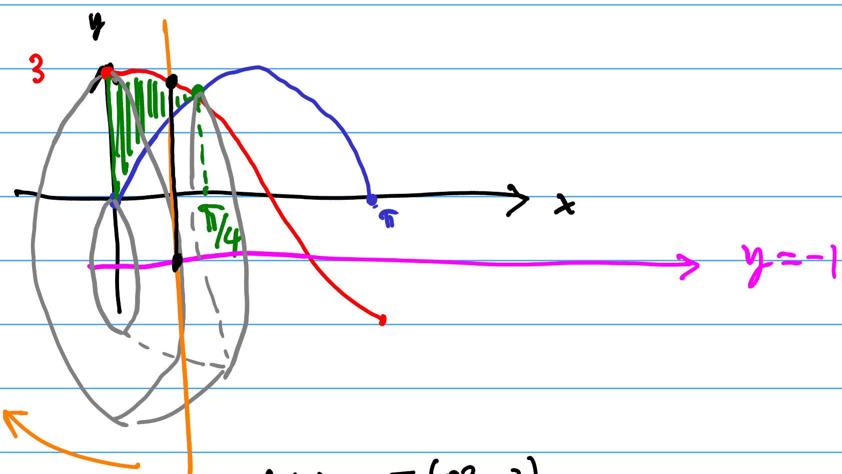
$$= \pi(18-0) - \pi\left(\tfrac{8}{3}-\cancel{8}+\cancel{8}\right)$$

$$= \left(18-\tfrac{8}{3}\right)\pi = \frac{46\pi}{3}$$

Ex. $\begin{cases} y = 3\sin x \\ y = 3\cos x \end{cases}$ Rotate about $y=-1$

$0 \le x \le \pi/4$





$R = 1+3\cos x$

$r = 1+3\sin x$ $\longrightarrow$ $A(x) = \pi(R^2-r^2)$

$$A(x) = \pi\left((1+3\cos x)^2 - (1+3\sin x)^2\right)$$

$$V = \pi\int_0^{\pi/4} (1+6\cos x + 9\cos^2 x) - (1 + 6\sin x + 9\sin^2 x)\,dx$$

$$= \pi\int_0^{\pi/4} 6\cos x - 6\sin x + 9(\underbrace{\cos^2 x - \sin^2 x}_{\cos(2x)})\,dx$$

$u = 2x \quad u(\pi/4) = \pi/2$

$du = 2dx \quad u(0) = 0$

Double Angle Formulas:

$\sin(2x) = 2\sin x\cos x$

$$\cos(2x) = \begin{cases} 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \\ \cos^2 x - \sin^2 x \end{cases}$$

$$= 6\pi\int_0^{\pi/4}\cos x\,dx - 6\pi\int_0^{\pi/4}\sin x\,dx + \frac{9\pi}{2}\int_0^{\pi/4} 2\cos(2x)\,dx$$

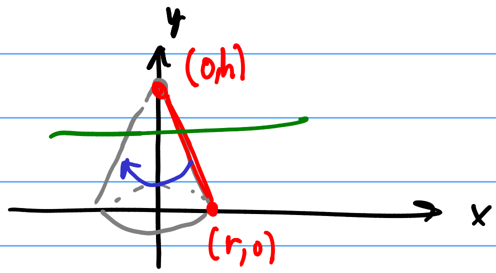
$$= 6\pi\sin x\Big|_0^{\pi/4} + 6\pi\cos x\Big|_0^{\pi/4} + \frac{9\pi}{2}\int_0^{\pi/2}\cos u\,du$$

$$= 6\pi\sin x\Big|_0^{\pi/4} + 6\pi\cos x\Big|_0^{\pi/4} + \frac{9\pi}{2}\sin u\Big|_0^{\pi/2}$$

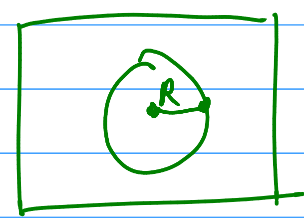
$$6\pi\left(\frac{\sqrt{2}}{2} - 0\right) + 6\pi\left(\frac{\sqrt{2}}{2} - 1\right) + \frac{9\pi}{2}(1-0)$$

$$\frac{\pi}{2}(12\sqrt{2} - 6 + 9) = \boxed{\frac{\pi(12\sqrt{2}+3)}{2}}$$

Ex. Compute the volume of a right circular cone:



Compute!



$\pi R^2 = \pi x^2$