MZ42 31 Oct 2018

§5.2 Volumes of Solids of Revolution – Examples

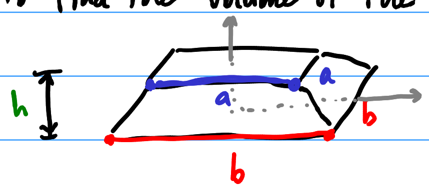
Ex. Find the volumes of the solids described below:

1. Rotate the region bounded between $y=2\sqrt{49-x^2}$, $y=0$, $x=3$, $x=6$ about the x-axis.

2. Rotate the region bounded between $y=x^2$ and $x=y^2$ about the line $y=1$.

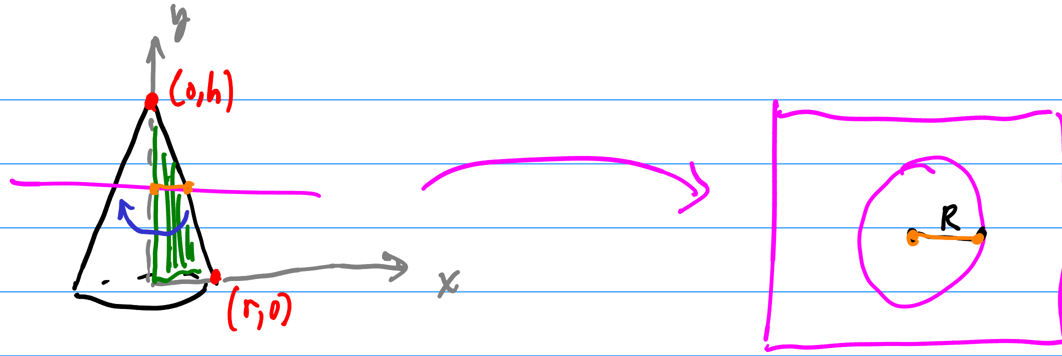
3. Use the slicing method to compute the volume of a sphere (ball) w/ radius $R>0$.

4. Use the slicing method to find the volume of the square "frustum":



$0<a<b$

Ex.



$$y = mx + b$$

$$m = -\frac{h}{r} \qquad b = h$$

$$y = -\frac{h}{r}x + h$$

$$y - h = -\frac{h}{r}x$$

$$x = \boxed{-\frac{r}{h}y + r = R}$$

$$A(y) = \pi R^2$$

$$V = -\frac{h}{r}\pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 \underbrace{-\frac{r}{h}\,dy}_{du}$$

$u = -\frac{r}{h}y + r \qquad u(h) = 0$

$du = -\frac{r}{h}\,dy \qquad u(0) = r$

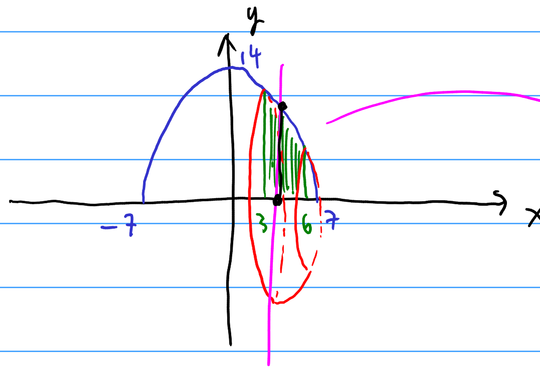
$$V = -\frac{h}{r}\pi \int_r^0 u^2\,du = \frac{h}{r}\pi \int_0^r u^2\,du$$

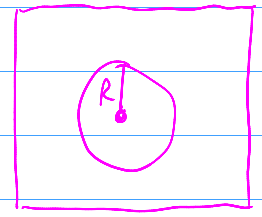
$$= \frac{h}{r}\pi \cdot \frac{1}{3}u^3 \Big|_0^r$$

$$= \frac{h}{r}\pi \frac{r^3}{3} = \frac{1}{3}\pi r^2 h \quad \checkmark$$

Brief Solutions to the exercises ↑

1. Rotate the an bounded between $y=2\sqrt{49-x^2}$, $y=0$, $x=3$, $x=6$ about the x-axis.





$$A=\pi R^2$$
$$=\pi\left(2\sqrt{49-x^2}\right)^2$$

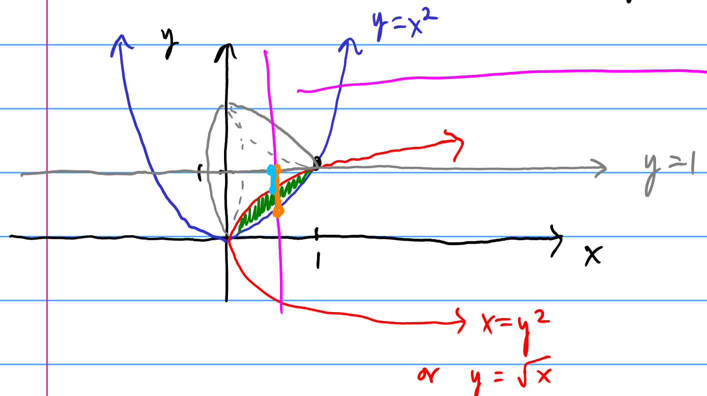
$$V=\int_3^6 \pi\left(2\sqrt{49-x^2}\right)^2 dx$$

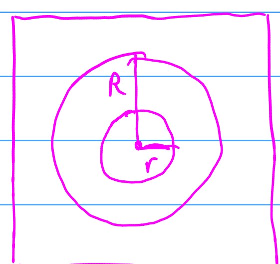
$$=4\pi\int_3^6 49-x^2\,dx$$

$$=4\pi\left(49x-\tfrac{1}{3}x^3\right)\Big|_3^6=4\pi\left(49(6-3)-\tfrac{1}{3}(6^3-3^3)\right)$$
$$=4\pi\left(147-(72-9)\right)$$
$$=4\pi(147-63)$$
$$=4\cdot 84\pi$$
$$=336\pi$$

---

2. Rotate the region bounded between $y=x^2$ and $x=y^2$ about the line $y=1$.





$R=1-x^2$

$r=1-\sqrt{x}$

$A=\pi\left(R^2-r^2\right)$

$$V=\pi\int_0^1 (1-x^2)^2-(1-\sqrt{x})^2\;dx$$

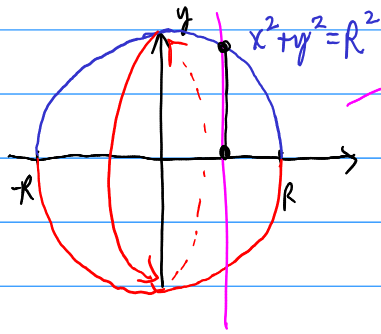
$$=\pi\int_0^1 \cancel{1}-2x+x^2-\left(\cancel{1}-2\sqrt{x}+x\right)\;dx$$

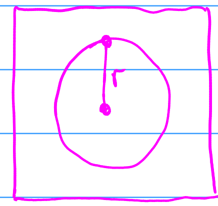
$$=\pi\int_0^1 x^2-3x+2\sqrt{x}\;dx$$

$$=\pi\left(\tfrac{1}{3}x^3-\tfrac{3}{2}x^2+2\cdot\tfrac{2}{3}x^{3/2}\right)\Big|_0^1$$

$$=\pi\left(\tfrac{1}{3}-\tfrac{3}{2}+\tfrac{4}{3}\right)=\pi\left(\tfrac{5}{3}-\tfrac{3}{2}\right)=\pi/6$$

3. Use the slicing method to compute the volume of a sphere (ball) w/ radius $R>0$.





$$A = \pi r^2 = \pi (y)^2 = \pi\left(\sqrt{R^2-x^2}\right)^2$$

$$V = \int_{-R}^{R} \pi\,(R^2-x^2)\,dx = 2\pi \int_0^R R^2 - x^2\,dx$$

$$= 2\pi\left(R^2x - \tfrac{1}{3}x^3\right)\Big|_0^R$$

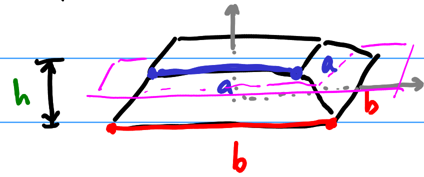
$$= 2\pi\left(R^3 - \tfrac{1}{3}R^3\right)$$

$$= 2\pi\left(\tfrac{2}{3}R^3\right)$$

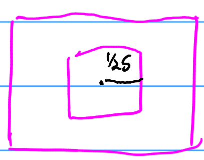
$$= \frac{4}{3}\pi R^3$$

---

4. Use the slicing method to find the volume of the square "frustum":

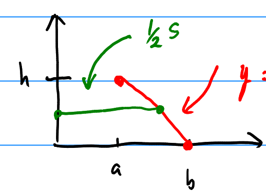


$0<a<b$



$A = s^2 = (2x)^2$

In xy-plane:



$y = \frac{-h}{b-a}(x-b)$ or $x = \frac{a-b}{h}y + b$

So, $V = \int_0^h 4\left(\frac{a-b}{h}y+b\right)^2 dy = 4\int_0^h \left(\frac{a-b}{h}\right)^2 y^2 + \frac{2(a-b)b}{h}y + b^2\,dy$

$$= 4\left(\frac{1}{3}\left(\frac{a-b}{h}\right)^2 y^3 + \frac{(a-b)b}{h}y^2 + b^2 y\right)\Big|_0^h$$

$$= 4\left(\tfrac{1}{3}(a-b)^2h + (a-b)bh + b^2h\right)$$

$$= 4h\cdot\left(\tfrac{1}{3}(a^2-2ab+b^2) + ab - \cancel{b^2} + \cancel{b^2}\right)$$

$$= \boxed{\frac{4}{3}(a-b)^2h + 4abh}$$