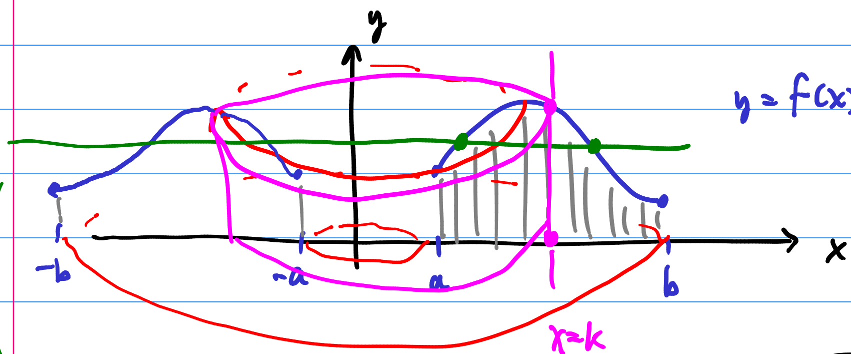
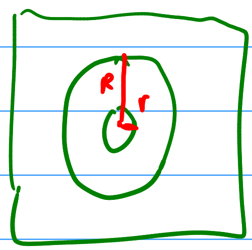
M242 — 1 Nov 18

§5.3 - Cylindrical Shell Method



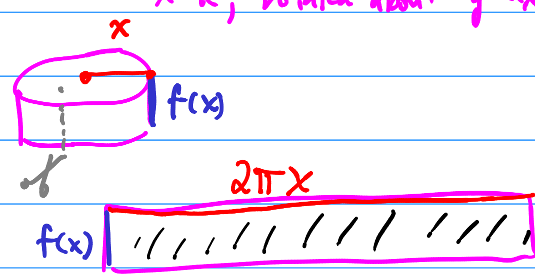
Goal: Find the volume.



$$A = \pi(R^2 - r^2)$$
$$= \pi\left(R(y)^2 - r(y)^2\right)$$
Difficult!

Instead

$x=k$, rotated about y-axis:



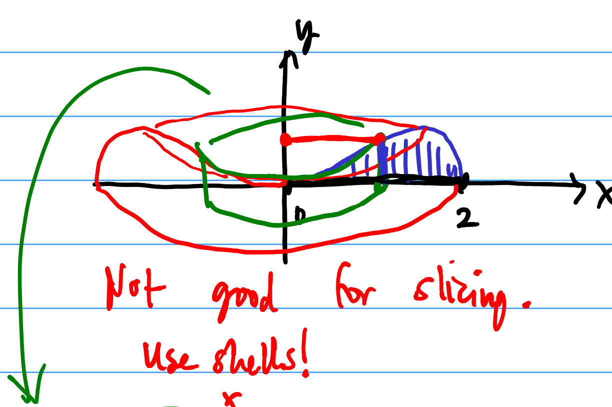
$$A(x) = 2\pi x f(x)$$

$$\boxed{V = \int_a^b 2\pi x\, f(x)\, dx} \quad (*) \leftarrow$$

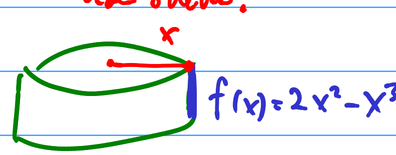
In general,
$$V = \int_a^b 2\pi R(x) H(x)\, dx$$
$R(x)$ = radius of shell
$H(x)$ = height of shell

Ex. $\begin{cases} y = 2x^2 - x^3 \\ y = 0 \end{cases}$

Set equal: $2x^2 - x^3 = 0$
$x^2(2-x) = 0$
$x = 0, \quad x = 2$



Not good for slicing.
Use shells!



$$A(x) = 2\pi x(2x^2 - x^3)$$

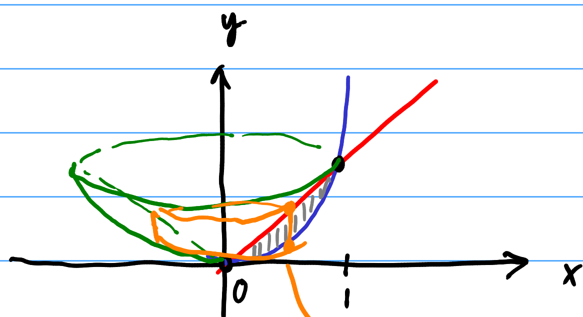
$$V = \int_0^2 2\pi\, x\,(2x^2 - x^3)\,dx$$

$$= 2\pi \int_0^2 2x^3 - x^4\,dx = 2\pi\left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Big|_0^2$$

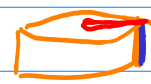
$$= 2\pi\left(8 - \frac{32}{5}\right)$$

$$= \frac{16\pi}{5}$$

Ex. $\begin{cases} y = x \\ y = x^2 \end{cases}$ rotate about the y-axis



slicing will work.
we'll use shells.



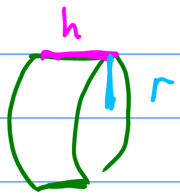
$r = x$
$h = (x - x^2)$

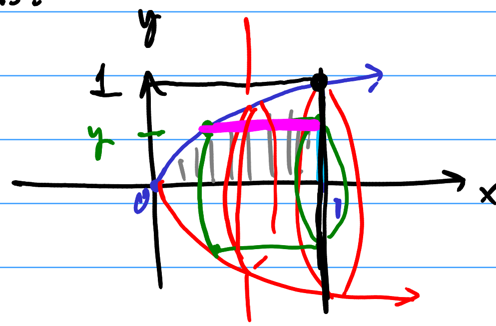
$A = 2\pi r h$
$= 2\pi\, x\,(x - x^2)$

$$V = \int_0^1 2\pi\, x(x - x^2)\,dx$$

$$= 2\pi \int_0^1 x^2 - x^3\,dx = 2\pi\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = \pi/6$$

Ex. $\begin{cases} y^2 = \sqrt{x}^2 \to x = y^2 \\ x: 0 \to 1 \end{cases}$ rotate about x-axis.
Use shells.



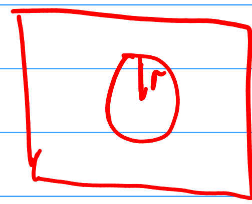


$r = y$
$h = 1 - y^2$

$$V=\int_0^1 2\pi y(1-y^2)\,dy$$

$$= 2\pi\int_0^1 y-y^3\,dy = 2\pi\left(\tfrac{1}{2}y^2-\tfrac{1}{4}y^4\right)\Big|_0^1 = \pi/2$$

check w/ slicing:



$$A=\pi r^2=\pi(\sqrt{x})^2$$

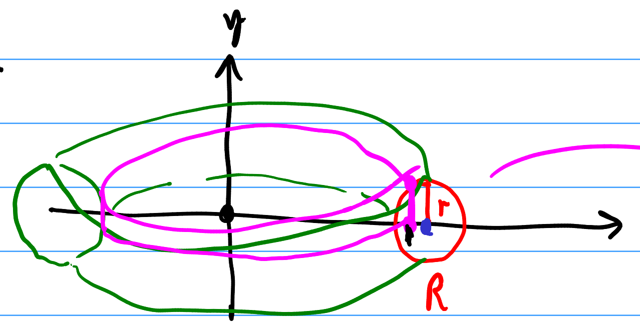
$$V=\int_0^1 \pi x\,dx = \pi\,\tfrac{1}{2}x^2\Big|_0^1 = \pi/2$$

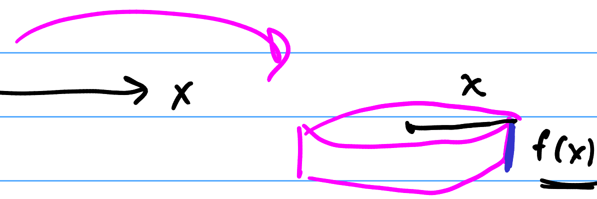
Ex. Torus

Bagel!

$0<r<R$

$x: R-r \longrightarrow R+r$





circle centered at $C(R,0)$
w/ radius r

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-R)^2+y^2=r^2$$

$$f(x)=y=\sqrt{r^2-(x-R)^2}$$

so, $$\tfrac{1}{2}V=\int_{R-r}^{R+r} 2\pi x\sqrt{r^2-(x-R)^2}\,dx$$

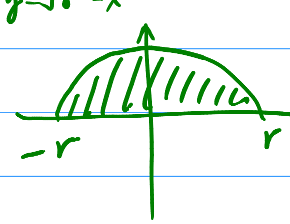
$u=x-R \rightarrow x=u+R$ $\qquad$ $u(R+r)=R+r-R=r$

$du=dx$ $\qquad$ $u(R-r)=R-r-R=-r$

$$V=2\int_{-r}^{r} 2\pi(u+R)\sqrt{r^2-u^2}\,du$$

$$=4\pi\left[\underbrace{\cancel{\int_{-r}^{r} u\sqrt{r^2-u^2}\,du}}_{0} + R\underbrace{\int_{-r}^{r}\sqrt{r^2-u^2}\,du}_{\frac{1}{2}\pi r^2}\right]$$

$y=\sqrt{r^2-x^2}$



$$f(u)=u\sqrt{r^2-u^2}$$

$$f(-u)=(-u)\sqrt{r^2-(-u)^2}=-\left(u\sqrt{r^2-u^2}\right)=-f(u)$$

$$= 4\pi \cdot \frac{1}{2} \pi r^2 R$$

$$\boxed{= 2\pi R \cdot \pi r^2}$$