§5.2–5.3: Volumes of Solids of Revolution – In-class Exercises

1.–5. For each solid described below: a) Sketch the region and the axis of rotation
b) Choose a method (slicing or shells) and sketch a typical section
c) Write down the integral for the volume for part c. and compute!

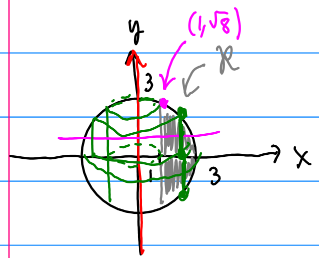
1. Rotate the region inside $x^2+y^2=9$, $x \geq 1$, about the y-axis.
2. Rotate the region bounded between $y=1+\sec x$ and $y=3$ about the line $y=1$.
3. Rotate the region bounded between $x=y^2$ and $x=1-y^2$ about the line $x=3$.
4. The cap of a sphere w/ radius r and height h.
5. Rotate the region bounded between $x=\sqrt{\sin y}$, $0 \leq y \leq \pi$, and $x=0$ about the line $y=4$.

6.–8. Describe the solid whose volume is given by the integral.

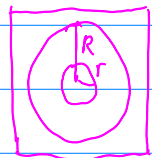
6. $\int_0^{\pi/2} 2\pi(x+1)(2x-\sin x)\,dx$ 7. $\pi\int_{-1}^{1}(1-y^2)^2\,dy$ 8. $2\pi\int_1^4 \frac{y+2}{y^2}\,dy$

Brief Solutions

1. Rotate the region inside $x^2+y^2=9$, $x \geq 1$, about the y-axis.



Slicing:



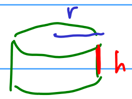
$R = x = \sqrt{9-y^2}$

$r = 1$

$A(y) = \pi\left((\sqrt{9-y^2})^2 - 1\right) = \pi(8-y^2)$

$$V = \pi\int_{-\sqrt{8}}^{\sqrt{8}} 8-y^2\,dy = 2\pi\int_0^{2\sqrt{2}} 8-y^2\,dy = 2\pi\left(8y-\tfrac{1}{3}y^3\right)\Big|_0^{2\sqrt{2}} = 2\pi\left(16\sqrt{2}-\tfrac{1}{3}\cdot 8\cdot 2\sqrt{2}\right) = \frac{4\cdot 16\pi\sqrt{2}}{3} = \frac{64\sqrt{2}\pi}{3}$$

Shells:



$r = 2\pi x$

$h = 2y = 2\sqrt{9-x^2}$

$A(x) = 4\pi x\sqrt{9-x^2}$

$$V = 4\pi\int_1^3 x\sqrt{9-x^2}\,dx = \left\{\begin{matrix} u=9-x^2 & u(3)=0 \\ du=-2x\,dx & u(1)=8 \end{matrix}\right\} = -2\pi\int_8^0 \sqrt{u}\,du = 2\pi\int_0^8 \sqrt{u}\,du = 2\pi\tfrac{2}{3}u^{3/2}\Big|_0^8 = \frac{4\pi}{3}\cdot\sqrt{8}^3 = \frac{64\sqrt{2}\pi}{3}$$

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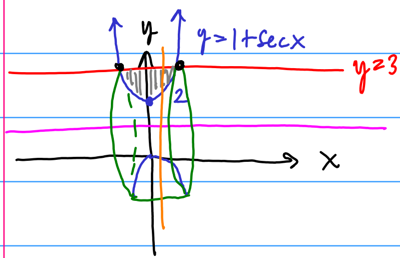
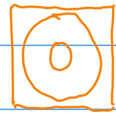
2. Rotate the region bounded between $y=1+\sec x$ and $y=3$ about the line $y=1$.



Slicing:



$R=2$

$r=\sec x$

$A(x)=\pi\left(4-\sec^2 x\right)$

$$V=\int_{-\pi/3}^{\pi/3}\pi\left(4-\sec^2x\right)dx = 2\pi\int_0^{\pi/3}4-\sec^2x\,dx = 2\pi\left(4x-\tan x\right)\Big|_0^{\pi/3} = \boxed{2\pi\left(\frac{4\pi}{3}-\sqrt{3}\right)}$$

$1+\sec x = 3$

$\sec x = 2$

$\cos x = 1/2$

$x=\pm\pi/3$

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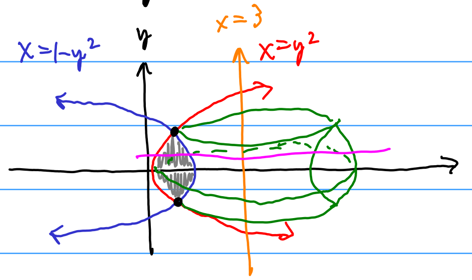
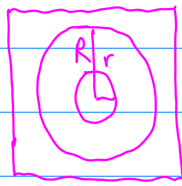
3. Rotate the region bounded between $x=y^2$ and $x=1-y^2$ about the line $x=3$.



Slicing:



$R=3-y^2$

$r=3-\left(1-y^2\right)=2+y^2$

$A(y)=\pi\left(\left(3-y^2\right)^2-\left(2+y^2\right)^2\right)$

$=\pi\left(9-6y^2+\cancel{y^4}-\left(4+4y^2+\cancel{y^4}\right)\right)$

$=\pi\left(5-10y^2\right)$

$1-y^2=y^2$

$y^2=1/2 \qquad y=\pm\frac{\sqrt2}{2}$

$$V=\int_{-\frac{\sqrt2}{2}}^{\frac{\sqrt2}{2}}\pi\left(5-10y^2\right)dy = 2\pi\int_0^{\sqrt2/2}5-10y^2\,dy = 2\pi\left(5y-\frac{10}{3}y^3\right)\Big|_0^{\sqrt2/2} = 2\pi\left(\frac{6\cdot5\sqrt2}{2\cdot6}-\frac{10}{3}\cdot\frac{\cancel{2}\sqrt2}{\cancel{8}4}\right)=2\pi\left(\frac{30\sqrt2-10\sqrt2}{12}\right)$$

$$=\frac{40\sqrt2\,\pi}{12}=\frac{10\sqrt2\,\pi}{3}$$

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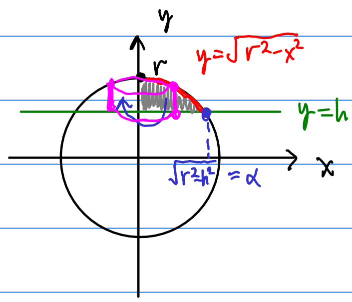
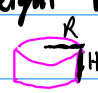
4. The cap of a sphere w/ radius r and height h.



Shells:



$R=x$

$H=y-h=\sqrt{r^2-x^2}-h$

$A(x)=2\pi x\left(\sqrt{r^2-x^2}-h\right)=2\pi x\sqrt{r^2-x^2}-2\pi hx$

$$V=\pi\int_0^\alpha \underbrace{2x}_{(-du)}\underbrace{\sqrt{r^2-x^2}}_{u}\,dx-2\pi h\int_0^\alpha x\,dx$$

$u(\alpha)=r^2-\left(r^2-h^2\right)=h^2$

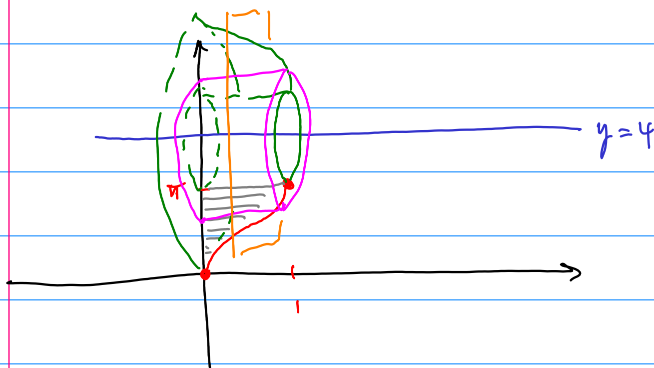
$u(0)=r^2$

$$V=\pi\int_{r^2}^{h^2}-\sqrt{u}\,du-2\pi h\left(\frac12\sqrt{r^2-h^2}^{\,2}\right)=\pi\left(\frac23u^{3/2}\Big|_{h^2}^{r^2}\right)-\pi hr^2+\pi h^3$$

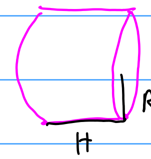
$$=\frac23\pi r^3-\frac23\pi h^3-\pi hr^2+\pi h^3$$

$$=\frac23\pi r^3+\frac13\pi h^3-\pi hr^2$$

5. Rotate the region bounded between $x=\sqrt{\sin y}$, $0 \leq y \leq \pi$, and $x=0$ about the line $y=4$.



Shells :



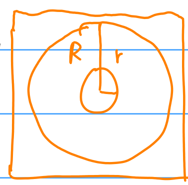
$R = 4-y$
$H = x = \sqrt{\sin y}$

$V(x) = 2\pi \int_0^{\pi} (4-y)\sqrt{\sin y}\, dy$

$= 8\pi \int_0^{\pi} \sqrt{\sin y}\, dy - 2\pi \int_0^{\pi} y\sqrt{\sin y}\, dy$

Can't do this integral ☹

Slicing:



$R = 4-y = 4-\arcsin(x^2)$

$r = 4-\pi$

$A(x) = \pi\left((4-\arcsin(x^2))^2 - (4-\pi)^2\right)$

$V = \pi \int_0^1 (4-\arcsin(x^2))^2 - (4-\pi)^2\, dx$ Can't do this integral either ☹

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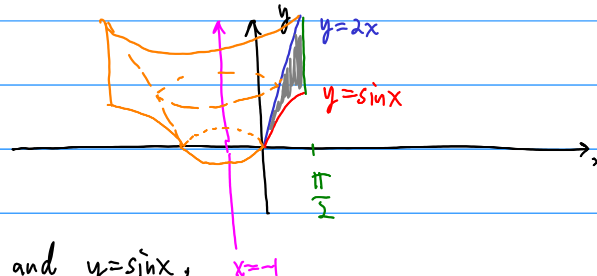
6. $\int_0^{\pi/2} 2\pi \underbrace{(x+1)}_{R} \underbrace{(2x-\sin x)}_{H} dx$

Shells!



$R = x+1$, so $x=-1$ is the axis of rotation

$H = 2x - \sin x$, so the region is bounded between $y=2x$ and $y=\sin x$, between $x=0$ and $x=\pi/2$.

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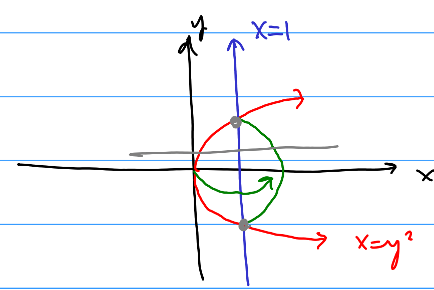
7. $\pi \int_{-1}^{1} (\underbrace{1-y^2}_{})^2\, dy$

Slicing! R^2

$R = 1-y^2$

axis is $x=1$ and the function is $x=y^2$



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8. $2\pi \int_1^4 \frac{y+2}{y^2}\, dy$

Shells!

$R = y+2$, $H = \frac{1}{y^2}$

axis is $y=-2$

$y: 1 \to 4$