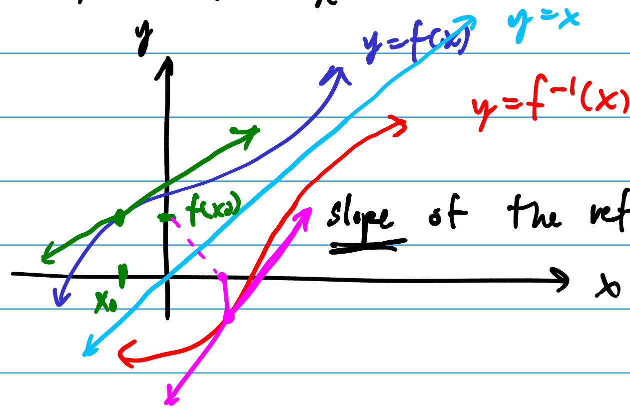
§6.1 Inverse Functions

Let f be a continuous, differentiable, 1-1 function such that $f'(x_0) \neq 0$.



slope of the reflected line is $(f^{-1})'(f(x_0))$

Thm. Let f be a differentiable 1-1 function w/ inverse function f^{-1}. Then the derivative of f^{-1} is given by

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$

as long as $f'(x_0) \neq 0$.

Proof. Suppose $f'(x) \neq 0$ or $dy/dx \neq 0$.

$$f^{-1}(x) = y \quad \underline{\text{means}} \quad x = f(y)$$

Take $\frac{d}{dx}$ implicitly:

$$\frac{d}{dx}\left[f(y) = x\right]$$

$$f'(y) \cdot \frac{dy}{dx} = 1, \text{ so}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

Notation is confusing (bad, Justin) ✓ ∎

Another POV:

If $f'(x) = \frac{dy}{dx}$, then $f^{-1}(f(x)) = \frac{dx}{dy}$

and $\frac{dx}{dy} = \frac{1}{dy/dx}$.

Ex. $f(x) = 2x + \cos x$ Find $(f^{-1})'(\underset{\boxed{y}}{1})$.

$(f^{-1})'(1) = \frac{1}{f'(?)}$ ← what is x?

$\neq 0$?

$f'(x) = 2 - \sin x$ and $-1 \leq \sin x \leq 1$

so $f'(x) \geq 1$ hence $f'(x) \neq 0$.] Tells us that f is 1-1.

To find x: solve $2x + \cos x = 1$

$2 \cdot 0 + \cos 0 = 0 + 1 = 1$ ✓

So, $(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$.

Ex. Find $(f^{-1})'(0)$ for $f(x) = \int_3^x \sqrt{1+t^3}\,dt$.

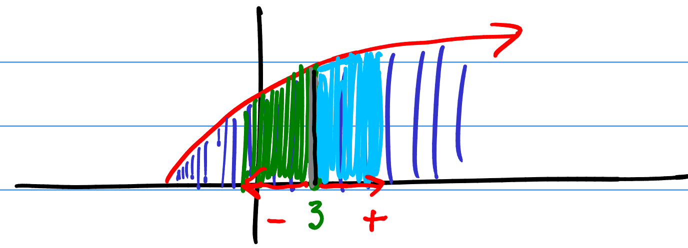
$(f^{-1})'(0) = \frac{1}{f'(?)}$ provided f is 1-1.

So, $f'(x) = \frac{d}{dx}\int_3^x \sqrt{1+t^3}\,dt = \sqrt{1+x^3}$ $\boxed{x \geq -1}$

$f'(x) \geq 0$ for $x \geq -1$ 1-1 $f'(x) \neq 0$!

Solve $f(x) = 0$ for x

$$\int_{3}^{x} \sqrt{1+t^3}\,dt = 0$$



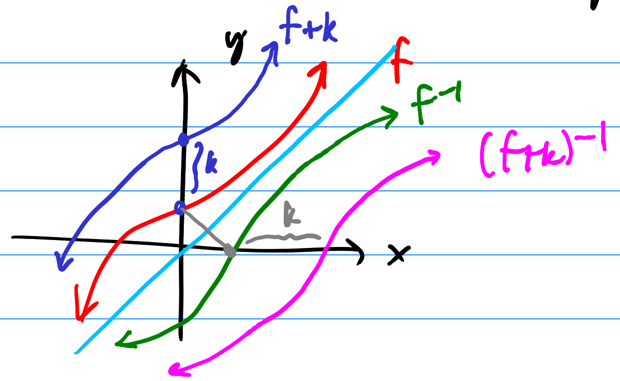
so $f(3)=0$

$$f(3) = \int_3^{3} \sqrt{1+t^3}\,dt = 0.$$

So $$(f^{-1})'(0) = \frac{1}{f'(3)} = \frac{1}{\sqrt{1+3^3}} = \frac{1}{\sqrt{28}}$$

Ex. let $y=f(x)$ be a H curve.

Consider the curve $y = f(x)+k$ for some constant k.



if $g(x) = f(x)+k$,

then $g^{-1}(x) = f^{-1}(x-k)$

Ex. $f(x) = \sqrt{x}$ and $k=4$ $\qquad f^{-1}(x) = x^2, \quad x \geq 0$

$g(x) = \sqrt{x}+4$

$$g^{-1}(x) = (x-4)^2 \qquad x \geq 0$$

0. $y = \sqrt{x} + 4$

1. $x = \sqrt{y} + 4$

2. Solve for y...

✓