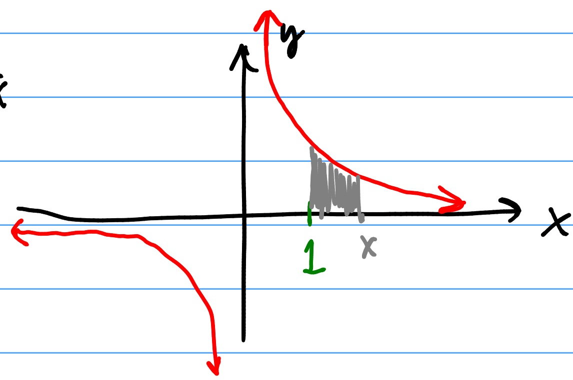
M242 — 14 Nov '18

§6.2* – Natural Logs

Motivation: $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$.

So consider $y = \frac{1}{x}$



$x > 0$

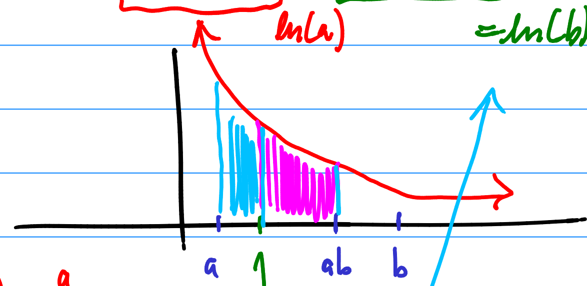
Define: $\boxed{\ln(x) = \int_1^x \frac{1}{t}\,dt \quad , x>0}$

Properties:

1.) $\ln(1) = 0$ because $\underbrace{\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0}$

$\boxed{2.)\ \ln(ab) = \ln(a) + \ln(b)}$ $\quad a, b > 0$

Proof. $\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \underbrace{\int_1^a \frac{1}{t}\,dt}_{\ln(a)} + \underbrace{\int_a^{ab} \frac{1}{t}\,dt}_{=\ln(b)\ ?}$



$\int_a^{ab} \frac{1}{t}\,dt = \begin{cases} u = \frac{t}{a} & u(a) = \frac{a}{a} = 1 \\ \underbrace{du = \frac{1}{a}\,dt} & u(ab) = \frac{ab}{a} = b \end{cases}$ $\rightarrow a\,du = dt$

$u = \frac{t}{a}$ $\quad \frac{1}{u} = \frac{a}{t}$

So, $\frac{1}{t} = \frac{1}{a}\cdot\frac{1}{u}$

$= \int_1^b \frac{1}{a}\frac{1}{u}\,a\cdot du = \int_1^b \frac{1}{u}\,du = \ln(b)$:)

$$\boxed{3.\ \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)}$$

Proof. FTIS.

$$\ln\left(\frac{a}{b}\right) = \int_1^{a/b} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \underbrace{\int_a^{a/b} \frac{1}{t}\,dt}_{= -\ln b\,?}$$

Ex. $\ln(x) = \int_1^x \frac{1}{t}\,dt$

$$(\ln(x))' = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

4. $\ln(x^r) = \underline{r\ln(x)}$

$$\frac{d}{dx}\left(\ln(x^r)\right) = \frac{d}{dx}\left(\ln(u)\right) = \frac{1}{u}\cdot u' = \frac{1}{x^r}\cdot r x^{r-1}$$

$$\boxed{\frac{d}{dx}\left(\ln(x^r)\right) = r\,\frac{1}{x}}$$

$$\frac{d}{dx}\left(r\ln(x)\right) = r\,\frac{1}{x}$$

So, $\ln(x^r) = r\ln(x) + C$ where C = constant

$x=1$: $\ln(1^r) = r\ln(1) + C$

$$0 = r\cdot 0 + C$$

$$C = 0.$$

Ex. $\frac{d}{dx}\left[\ln(\underbrace{\cos x+2x}_{u})\right] = \frac{d}{dx}\left[\ln u\right] = \frac{1}{u}\cdot u' = \frac{u'}{u}$

$$\frac{d}{dx}\left[\ln(\cos x+2x)\right] = \frac{-\sin x+2}{\cos x+2x}$$

Ex. $\frac{d}{dx}\left[\ln\underbrace{\sqrt{x^2+1}}\right] = \frac{u'}{u} = \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} = \frac{x}{x^2+1}$

$u = \sqrt{x^2+1}$

$u' = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

OR $\frac{d}{dx}\left[\ln\sqrt{x^2+1}\right] = \frac{d}{dx}\left[\ln(x^2+1)^{1/2}\right] = \frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1)\right]$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}x}{x^2+1} = \frac{x}{x^2+1}$$

Ex. $\ln(x^2-1) = \ln\left((x-1)(x+1)\right) = \underbrace{\ln(x-1)+\ln(x+1)}$

$\frac{d}{dx}\left[\ln(x^2-1)\right] = \frac{u'}{u} = \underbrace{\frac{2x}{x^2-1}}$

OR

$\frac{d}{dx}\left[\ln(x^2-1)\right] = \frac{d}{dx}\left[\ln(x-1)+\ln(x+1)\right] = \underbrace{\frac{1}{x-1}+\frac{1}{x+1}}$

$$= \frac{(x+1)+(x-1)}{(x-1)(x+1)}$$

$$= \frac{2x}{x^2-1}$$

Ex. $\frac{1}{3}\int \frac{3 \cdot 4x^2}{x^3-2}\,dx$ $\quad u = x^3-2$

$du = 3x^2\,dx$

$= \frac{4}{3}\int \frac{1}{u}\,du = \frac{4}{3}\ln u + C$

$= \frac{4}{3}\ln\left|(x^3-2)\right| + C$

Ex. $\begin{cases} \frac{d}{dx}[\ln x] = \frac{1}{x} & \text{for } x>0 \\ \frac{d}{dx}[\ln(-x)] = \frac{u'}{u} = \frac{-1}{-x} = \frac{1}{x} & \text{for } x<0 \end{cases}$

So $\int \frac{d}{dx}\left[\ln|x|\right] = \int \frac{1}{x}\,dx$ for $x \neq 0$.

$\int \frac{1}{x}\,dx = \ln|x| + C$ most general antiderivative

Ex. $\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}\,dx$ $\quad u = \cos x$

$du = -\sin x\,dx$

$= -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C$

$= \ln\left|(\cos x)^{-1}\right| + C$

$= \ln\left|\frac{1}{\cos x}\right| + C$

$$\boxed{\int \tan x\,dx = \ln|\sec x| + C}$$

$$\boxed{\frac{dy}{dx} = y \qquad \text{Find } y.} \longleftarrow ?$$