M242 15 Nov '18

→ §6.2*, 6.3*

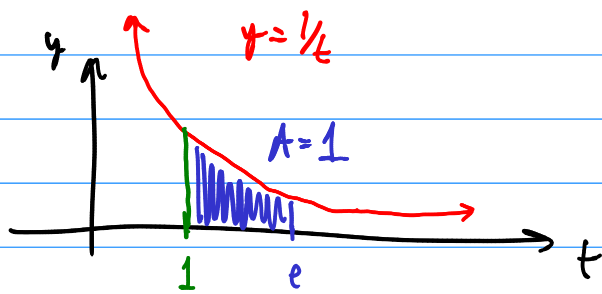
$\ln(x) = \int_1^x \frac{1}{t}\,dt \quad , x > 0$

Def'n. The natural number, e, is the number s.t.

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

e is irrational

$e \approx 2.7181828\ldots$



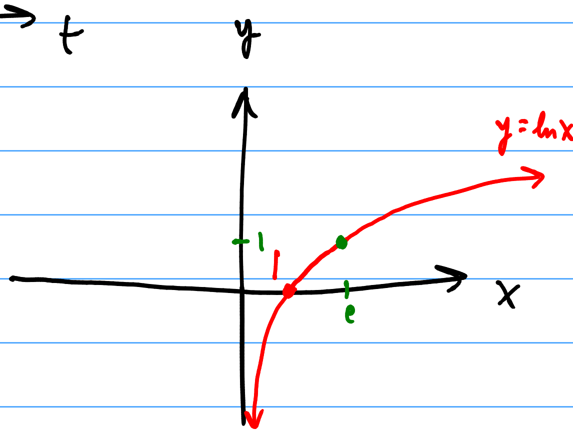
Graph $y = \ln x$

$\ln(1) = 0$

$\ln(e) = 1$

$\lim_{x\to\infty} \ln x = +\infty$

$\lim_{x\to 0^+} \ln x = -\infty$



the graph of $y = \ln x$ passes the HLT, and is therefore 1-1.
Thus it has an inverse function.

Def'n. The natural exponential function $y = \exp(x)$ is the inverse function of the natural log.

$$y = \exp(x) \iff x = \ln(y)$$

So, domain (exp) = range (ln) = $\mathbb{R}$

range (exp) = domain (ln) = $(0, \infty)$

In fact, $\exp(x) = e^x$ is an exponential function in algebra.

Properties:
$$\begin{cases} 1.\ e^{a+b} = e^a e^b \\ 2.\ e^{a-b} = \dfrac{e^a}{e^b} \\ 3.\ (e^a)^r = e^{ra} \end{cases}$$

4. $\ln(e^x) = x$ for all $x \in \mathbb{R}$
5. $e^{\ln x} = x$ for $x > 0$

Ex. Find a function s.t. $\frac{dy}{dx} = y$, $y(0) = 1$.

$$\frac{dy}{dx} = y \Rightarrow dy = y\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int dx$$
$$\ln|y| = x + C$$

plug in $y(0)=1$:

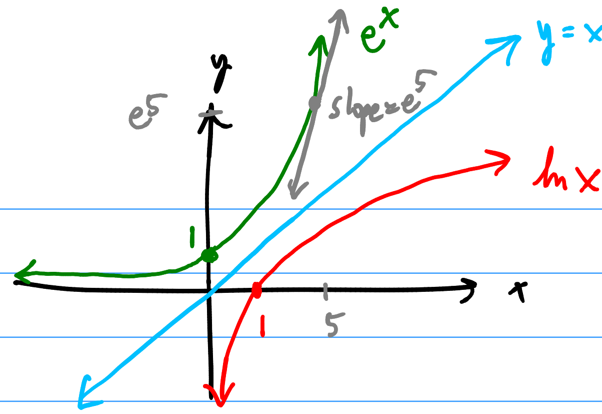
$$\ln(1) = 0 + C$$
$$\boxed{0 = C}$$

y satisfies $e^{\ln|y|} = e^x$

$$|y| = e^x$$

$$\boxed{y(x) = e^x \quad \text{satisfies} \quad \frac{dy}{dx} = y}$$

So, $\boxed{\frac{d}{dx}\left[e^x\right] = e^x \quad \text{and} \quad \int e^x\,dx = e^x + C}$



$e^0 = 1$ y-int is $y=1$

$\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to\infty} e^x = \infty$

Ex. $f(x) = e^{2x^2}$ u

$f'(x) = f'u' = u'e^u = 4x\,e^{2x^2}$

Ex. $g(x) = \sin x\, e^{\cos x}$

$\int g(x)\,dx = -\int -\sin x\, e^{\cos x}\,dx = -\int e^u\,du = -e^u + C = -e^{\cos x} + C$

$u = \cos x$
$du = -\sin x\,dx$

Ex. $\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = L\right)$

$y = \ln(x)$ is 1-1
and
$y = \ln(x)$ is continuous

$\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \ln L$

$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = \ln L$

$\lim_{x\to\infty} x\ln(1+1/x) = \infty\cdot\ln(1) = \infty\cdot 0$

= Now time jump to Calc II =

$= \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = \frac{0}{0} \overset{\boxed{L'H}}{=} \lim_{x\to\infty} \frac{(\ln(1+1/x))'}{(1/x)'}$

$(\ln u)' = u'\cdot\frac{1}{u}$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot \cancel{\left(-1/x^2\right)}}{\cancel{-1/x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+1/x} = \frac{1}{1+0} = 1$$

so, $e^{\ln L} = e^{1} \Rightarrow L = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$ (*)

<u>A third defn</u>. e is the number satisfying

$$\underbrace{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$$

<u>Fact</u>. all Use this fact to compute $(e^x)'$ via the defn.

$$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= e^x \left(\underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{=1} \right)$$

$$= e^x \cdot 1$$

$$= e^x \quad \checkmark$$