M242 19 Nov 18

3. Find the function $y=y(x)$ that satisfies the initial value problem.

a.) $\begin{cases} dy/dx = 3y \\ y(0)=1 \end{cases}$ ⟸ b.) $\begin{cases} x\frac{dy}{dx}=1 \\ y(1)=3 \end{cases}$ c.) $\begin{cases} dy/dx = 2y-3 \\ y(0)=0 \end{cases}$

a.) $\frac{dy}{\cancel{dx}} \cancel{dx} = 3y\, dx \Rightarrow \frac{1}{y}\, dy = \frac{3y}{y}\, dx \Rightarrow \int \frac{1}{y}\, dy = \int 3\, dx$

$\ln|y| = 3x + C \xrightarrow{(0,1)} \ln(1) = 3\cdot 0 + C$

$0 = 0 + C$ or $C = 0.$

$e^{\ln|y|} = e^{3x}$ $\boxed{y = e^{3x}}$

Check: $y = e^{3x}$ $y' = (e^u)' = e^u \cdot u' = e^{3x}\cdot 3 = 3y.$ ✓

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b.) $x\frac{dy}{dx} = 1 \longrightarrow x\,dy = dx \longrightarrow \int dy = \int \frac{1}{x}\, dx$

$y(1)=3$

$y = \ln|x| + C$

$\longrightarrow (1,3): \quad 3 = \cancel{\ln 1}^{\,0} + C$

$\boxed{y = \ln|x| + 3}$

$$\left.\begin{aligned} \frac{dy}{dx} &= 2y-3 \\ y(0) &= 0 \end{aligned}\right\}$$

$$\frac{dy}{dx} - 2y = -3$$

$$dy - \underbrace{2y\,dx} = -3dx$$

$$\frac{dy}{dx} = 2\left(y - \frac{3}{2}\right) \rightarrow \int \frac{1}{y-3/2}\,dy = \int 2\,dx$$

$u = y - 3/2$

$du = dy$

$$\int \frac{1}{u}\,du = \ln|u|$$

So, $\ln\left|y - \frac{3}{2}\right| = (2x + C_1)$

$e \qquad e$

$$y - \frac{3}{2} = e^{C_1} e^{2x}$$

$\uparrow$

k

$$y = Ke^{2x} + \frac{3}{2} \leftarrow$$

$y(0) = 0 \rightarrow (0,0)$

$\rightarrow 0 = Ke^0 + \frac{3}{2}$

$0 = K + 3/2$

$K = -\frac{3}{2}$

$$\boxed{y = -\frac{3}{2}e^{2x} + \frac{3}{2}}$$

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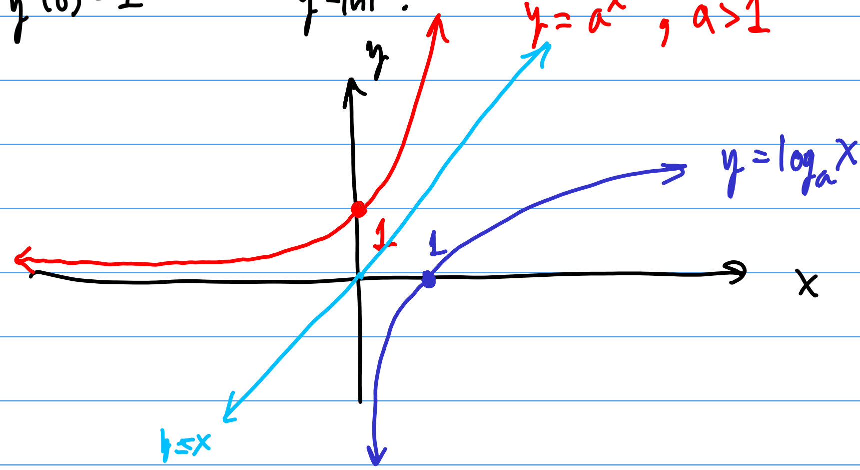
§ 6.4* Other bases

Let $a > 1$, and consider the exponential function $y = a^x$.

Sketch the graph. $a =$ "base" $\lim_{x\to\infty} a^x = +\infty$

$$\lim_{x \to -\infty} a^x = \lim_{u \to \infty} a^{-u} = \lim_{u \to \infty} \frac{1}{a^u} = 0$$

$a^0 = 1$ $\quad y(0) = 1 \quad$ y-int!



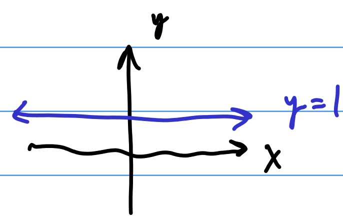
So, $y = a^x$, $a > 1$ is 1-1 and must have an inverse function.

Def'n. The logarithm w/ base a, $a > 1$, is the inverse function of $y = a^x$.

So, $y = \log_a(x)$ means $x = a^y$

Idea: $y = 1^x = 1$

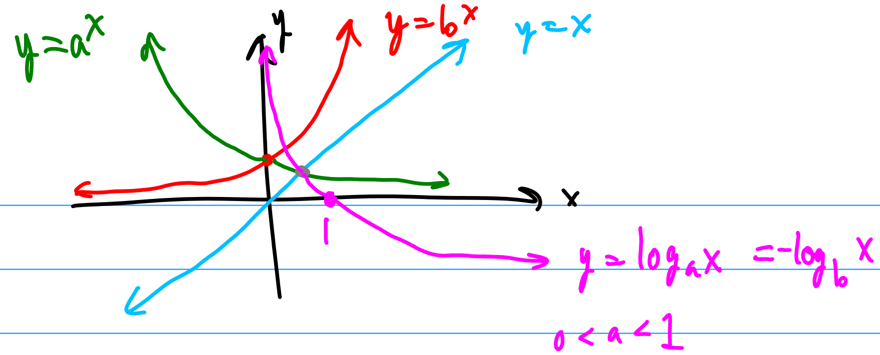
Not 1-1.



Other case, $\quad 0 < a < 1 \quad\quad y = a^x$

but if $0 < a < 1$, then $a = \frac{1}{b}$ for $b > 1$.

So $\quad y = a^x = b^{-x} \quad\quad b > 1$.



$y=a^x$ is 1-1.

<u>Ex.</u> $y=a^x \qquad 0<a, \ a\neq 1.$

Find y'. Apply inverse property:

$$e^{\ln a} = a \quad \text{so long as } a>0.$$

$$y=a^x = (e^{\ln a})^x = e^{x\cdot \ln a} = e^u$$

<u>Chain Rule!</u> $y' = u'e^u = \ln a \ \underbrace{e^{x\cdot\ln a}}_{a^x} = (\ln a)\, a^x$

so,

$$\boxed{\frac{d}{dx}\left[a^x\right] = (\ln a)\, a^x} \quad \text{for } a>0, \ a\neq 1.$$

Now, compute $\frac{dy}{dx}$ for $y=\log_a x$.

$y=\log_a x$ <u>means</u> $x=a^y$

Use implicit differentiation to get:

$$\frac{d}{dx}\left[x = a^y\right]$$

$$1 = (\ln a)\, a^y \cdot \frac{dy}{dx}$$

$$\boxed{\frac{d}{dx}\left[\log_a x\right] = \frac{1}{x \ln a}}$$