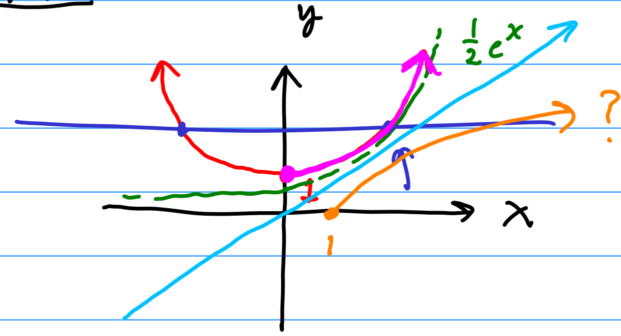
M242 30 Nov '18

§6.8 – Inverse Hyperbolic Functions



Ex. $y=\cosh(x)$

$y=\frac{1}{2}(e^x+e^{-x})$

restrict: $x \geq 0$ (0)

Now $\cosh(x)$ is 1-1.

R: $[1,\infty)$

$y=\cosh^{-1}(x) = \operatorname{arccosh}(x)$ means $\underbrace{\cosh(y)}=x$

$\frac{1}{2}(e^y+e^{-y})=x$ solve for y.

$e^y(e^y+e^{-y}=2x)$ $e^y>0$

$(e^y)^2+1=2xe^y$ ~~$\ln(e^y)=\ln(2x-e^{-y})$~~

$\underset{u}{(e^y)^2} \underbrace{-2x}\underset{u}{e^y} +1 = 0$ $\underbrace{u^2+bu+c=0}$

$b=-2x$

$\frac{b}{2}=-x \leftarrow$

$\left(\frac{b}{2}\right)^2=x^2 \leftarrow$

$u^2-2xu+x^2=-1+x^2$

$(u-x)^2=x^2-1$

$u-x=\pm\sqrt{x^2-1}$

$e^y=u=x\pm\sqrt{x^2-1} = x+\sqrt{x^2-1}$

↖ corresponds to left arm of graph

$$y = \boxed{\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)}$$

Ex. $y = \operatorname{arcsinh}(x)$ Find a formula.

$y = \sinh^{-1}(x)$

$$x = \sinh(y)$$
$$2x = e^y - e^{-y}$$
$$\rightarrow e^{2y} - 2xe^y - 1 = 0$$
$$e^y = x \pm \sqrt{x^2+1}$$
$$\ln e^y = \ln x + \sqrt{x^2+1}$$

$\left(\frac{b}{2}\right)^2 \quad \left(\frac{-2x}{2}\right)^2$

$-x^2$

$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$$\boxed{\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)}$$

Ex. $y = \operatorname{arctanh}(x) \rightarrow \tanh(y) = x$

$$x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)} = \frac{\cancel{\frac{1}{2}}(e^y - e^{-y})}{\cancel{\frac{1}{2}}(e^y + e^{-y})}$$

solve for y:

$$xe^y + xe^{-y} = 1e^y - e^{-y}$$

$$e^y\left(0 = (1-x)e^y - (1+x)e^{-y}\right)$$

$$0 = (1-x)e^{2y} - (1+x)$$

$$\frac{1+x}{1-x} = \frac{(1-x)e^{2y}}{1-x}$$

$$\ln\left(e^{2y} = \frac{1+x}{1-x}\right)$$

$$2y = \ln\left(\left|\frac{1+x}{1-x}\right|\right)$$

$$\operatorname{arctanh}(x) = y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \ln\sqrt{\frac{1+x}{1-x}}$$

$$(\ln u)' = \frac{1}{u}\cdot u'$$

$$= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|$$

Ex. $\frac{d}{dx}\left[\operatorname{arctanh}(x)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)\right]$

$$= \frac{1}{2}\left(\ln(1+x)\right)' - \frac{1}{2}\left(\ln(1-x)\right)'$$

$$= \frac{1}{2}\cdot\frac{1}{1+x}\cdot 1 + \frac{1}{2}\cdot\frac{1}{1-x}\cdot(1)$$

$$= \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{2}\left(\frac{1-x+1+x}{1-x^2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{1-x^2}\right) =$$

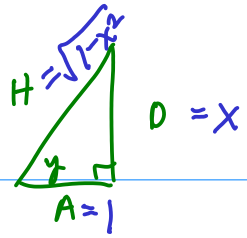
$$\boxed{\frac{d}{dx}\left[\operatorname{arctanh}(x)\right] = \frac{1}{1-x^2}} \quad (*)$$

Ex. $y = \arctan(x) \longrightarrow \tan(y) = x$ ...

$$(\arctan x)' = \frac{1}{1+x^2}$$

Ex. $y = \operatorname{arctanh}(x) \longrightarrow \tanh(y) = x$

take $\frac{d}{dx}$ implicitly: $\frac{d}{dx}\left[\tanh(y) = x\right]$



HPT: $A^2 - O^2 = H^2$ ← 11

$\tanh(y) = \frac{x}{1}$

$H^2 = 1^2 - x^2$

$H = \sqrt{1-x^2}$

$$\operatorname{sech}^2(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2(y)} = \cosh^2(y) = \left(\frac{1}{\sqrt{1-x^2}}\right)^2$$

so $\frac{dy}{dx} = \frac{1}{1-x^2}$! ☺

Ex. $y = \operatorname{arcsinh}(x)$ Find dy/dx.

$$y = \operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right) \qquad \sqrt{v}' = \frac{v'}{2\sqrt{v}}$$

$$y' = \frac{u'}{u} = \frac{1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}}{\frac{x+\sqrt{x^2+1}}{1}}$$

$$= \frac{\cancel{x+\sqrt{x^2+1}}}{\sqrt{x^2+1}} \cdot \frac{1}{\cancel{x+\sqrt{x^2+1}}} = \frac{1}{\sqrt{1+x^2}}$$

so $\boxed{\left(\operatorname{arcsinh}(x)\right)' = \frac{1}{\sqrt{1+x^2}}}$

On Final: $(8.5 \times 11)\ in^2$ both sides

$187\ in^2$ of handwritten notes