M242 Calc I — 6 Nov '18

Work Examples

5.4.31 The kinetic energy (KE) of an object w/ mass m and velocity v is $KE = \frac{1}{2}mv^2$. If a force $f = f(x)$ acts on the object moving it along the x-axis from x_1 to x_2, the <u>Work-Energy Theorem</u> states that the net work done is equal to the change of kinetic energy: $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, where $v_1 = v(x_1)$ and $v_2 = v(x_2)$.

a.) Let $x = s(t)$ be the position function of a particle, v the velocity, and a the acceleration. Prove the Work-Energy Theorem:

$$W = \int_{x_1}^{x_2} f(x)\,dx = \int_{t_1}^{t_2} f(s(t))\,d(s(t)) = \int_{t_1}^{t_2} \underline{f(s(t))} \cdot \underline{v(t)}\,dt$$

$v(t) = \frac{ds}{dt}$

$f(s(t)) = m\,a(t) = m\frac{dv}{dt}$

$$= \int_{t_1}^{t_2} m\,\underline{v(t)}\,\frac{dv}{dt}\,dt$$

$u = v(t)$

$du = \frac{dv}{dt}\,dt$

$$= \int_{t_1}^{t_2} m\,u\,du = \frac{1}{2}mu^2\Big|_{t_1}^{t_2} = \frac{1}{2}mv(s(t_2))^2 - \frac{1}{2}mv(s(t_1))^2$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$= KE(x_2) - KE(x_1) \quad \blacksquare$$

b.) How much work in ft-lbs is required to hurl a 12 lb bowling ball at 20 mph?

$$W = \Delta KE = \frac{1}{2}mv^2 = \frac{1}{2}\cdot\frac{3}{8}\cdot\left(\frac{88}{3}\right)^2 = \frac{11\cdot 88}{6} = \frac{11\cdot 44}{3} = \boxed{\frac{484}{3}\text{ ft-lbs}}$$

$$m = \frac{12\text{ lb}}{32\text{ ft/s}^2} = \frac{3}{8}\text{ slugs}$$

$$v = 2\cancel{0}\,\frac{\cancel{\text{mi}}}{\cancel{\text{h}}}\cdot\frac{528\cancel{0}\text{ ft}}{1\,\cancel{\text{mi}}}\cdot\frac{1\,\cancel{\text{h}}}{36\cancel{00}\text{ s}} = \frac{528}{18}\frac{\text{ft}}{\text{s}} = \frac{264}{9} = \frac{88}{3}\frac{\text{ft}}{\text{s}}$$

5.4.32 Suppose that when launching an 800 kg roller coaster car an electromagnetic propulsion system exerts a force of $f(x) = 5.7x^2 + 1.5x$ Newtons on the car at a distance of x meters along the track. Use the Work-Energy Theorem to find the speed of the car when it has traveled 60 m.

$$W = \int_0^{60} 5.7x^2 + 1.5x\,dx = \frac{5.7}{3}x^3 + \frac{1.5}{2}x^2\Big|_0^{60} = \frac{5.7}{3}60^3 + \frac{1.5}{2}60^2 = \left(5.7(20) + \frac{1.5}{2}\right)(3600)$$

$$= (114.75)(3600) = \frac{1}{2}mv^2, \quad \text{so } v = \sqrt{\frac{2(114.75)(3600)}{800}} = \boxed{32.14\text{ m/s}}$$

5.4.33 Newton's Law of Gravity: Two bodies w/ masses m_1 and m_2 attract each other with a force $F = \frac{m_1 m_2 G}{r^2}$

where $G = 6.67 \times 10^{-11} \frac{Nm^2}{Kg^2}$ is the gravitational constant and r is the distance between the centers of mass.

a.) If one of the bodies is fixed, how much work is done by gravity in moving the other body from $r=a$ to $r=b$?

$$W = \int_a^b F(r)\,dr = \int_a^b \frac{m_1 m_2 G}{r^2}\,dr = m_1 m_2 G \int_a^b \frac{1}{r^2}\,dr = m_1 m_2 G \left. -\frac{1}{r} \right|_a^b$$
$$= m_1 m_2 G \left(-\frac{1}{b} + \frac{1}{a}\right) = \boxed{m_1 m_2 G \left(\frac{b-a}{ab}\right) = W}$$

b.) Compute the work required to launch a 1000 kg satellite vertically to a height of 1000 km.

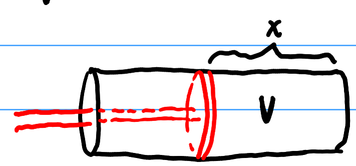
Earth's mass $= 5.98 \times 10^{24}$ kg

Earth's radius $= 6.37 \times 10^6$ m

$$W = m_1 m_2 G \left(\frac{b-a}{ab}\right) = (5.98 \times 10^{24})(1 \times 10^3)(6.67 \times 10^{-11}) \left(\frac{1 \times 10^6}{(1 \times 10^6)(6.37 \times 10^6)}\right)$$
$$= \frac{(5.98)(6.67)}{6.37} \times 10^{10} = \boxed{6.26 \times 10^{10} \text{ J}}$$

5.4.29. When gas expands in a cylinder w/ radius r, the pressure at any given time is a function of the volume, $P = P(V)$.

The force exerted by the gas on a piston is the product of the pressure and the area: $F = \pi r^2 P$.



The work done by the gas when the volume expands from V_1 to V_2 is:

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} \pi r^2 P(V(x))\,dx = \boxed{\int_{V_1}^{V_2} P(V)\,dV = W}$$

$dV = \pi r^2 dx$

$F(x) = \pi r^2 P(V)$

5.4.30. In a steam engine the pressure P and volume V of steam satisfy the equation $PV^{1.4}=k$, where $k=$ constant.

This is true for <u>adiabatic</u> expansion: expansion in which there is no heat transfer between the cylinder and the surroundings.

Calculate the work done by the engine during a cycle when the steam starts at $160\ \mathrm{lb/in^2}$ and a volume of $100\ \mathrm{in}^3$ and expands to a volume of $800\ \mathrm{in}^3$.

Work is measured in ft-lbs, so $P(V_0)=160\ \frac{\mathrm{lb}}{\mathrm{in}^2}\cdot\left(\frac{12\ \mathrm{in}}{\mathrm{ft}}\right)^2=(160)(144)\ \mathrm{lb/ft^2}$

$$V_0=100\ \mathrm{in}^3\left(\frac{\mathrm{ft}}{12\ \mathrm{in}}\right)^3=\frac{100}{12^3}\ \mathrm{ft}^3$$

$$V_1=800\ \mathrm{in}^3\left(\frac{\mathrm{ft}}{12\ \mathrm{in}}\right)^3=\frac{800}{12^3}\ \mathrm{ft}^3$$

$$PV^{1.4}=k=(160)(144)\left(\frac{100}{12^3}\right)^{1.4} \quad \text{and} \quad P(V)=kV^{-1.4}$$

$$W=\int_{V_0}^{V_1}P(V)\,dV=\int_{V_0}^{V_1}kV^{-1.4}\,dV=\left.\frac{kV^{-0.4}}{-0.4}\right|_{V_0}^{V_1}=-\frac{5}{2}(160)(144)\left(\frac{100}{12^3}\right)^{1.4}\left(\left(\frac{800}{12^3}\right)^{-0.4}-\left(\frac{100}{12^3}\right)^{-0.4}\right)$$

$$=-\frac{5}{2}(160)(144)\left(\frac{100}{12^3}\right)^{1.4}(12)^{1.2}\left(800^{-0.4}-100^{-0.4}\right)$$

$$\boxed{W=1882.42\ \text{ft-lbs}}$$