M243 8/23

§12.3 – Dot Product

Ex. $\vec{u} = \langle 2,4 \rangle$ $\quad \vec{v} = \langle 3,-1 \rangle$

$\vec{u}\cdot\vec{v} = 2(3) + 4(-1) = 6 - 4 = 2$

$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \|\vec{w}\| = \sqrt{\vec{w}\cdot\vec{w}}$

$\vec{w} = \vec{u}-\vec{v} = \langle -1, 5 \rangle$

$\vec{w}\cdot\vec{w} = (-1)^2 + 5^2 = 26$

$d(\vec{u},\vec{v}) = \|\vec{w}\| = \sqrt{26}$

Properties. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and c be a scalar.

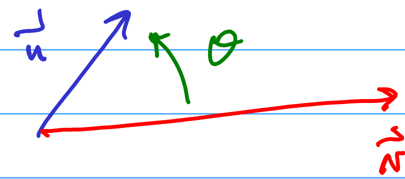
1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2. $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$
3. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
4. $(c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$
5. $\vec{0}\cdot\vec{u} = 0$

Summarizing, the dot product is linear:

$$(a\vec{u} + b\vec{v})\cdot\vec{w} = a(\vec{u}\cdot\vec{w}) + b(\vec{v}\cdot\vec{w})$$

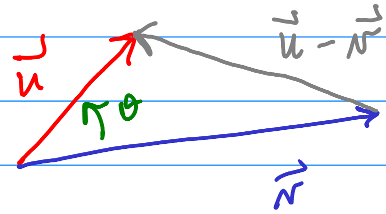
Def'n. Let $\vec{u}$ and $\vec{v}$ be vectors. The angle between $\vec{u}$ and $\vec{v}$ is the smallest non-negative angle between them.

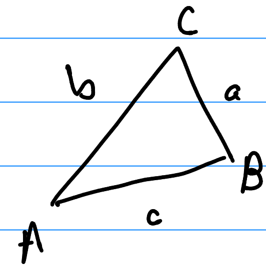
$0 \le \theta \le \pi$



$\vec{u}, \vec{v} \neq \vec{0}$

Q. Given two nonzero vectors $\vec{u}, \vec{v}$, how do we compute the angle?





Apply the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

or

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Plugging in, we get

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Now solve for θ

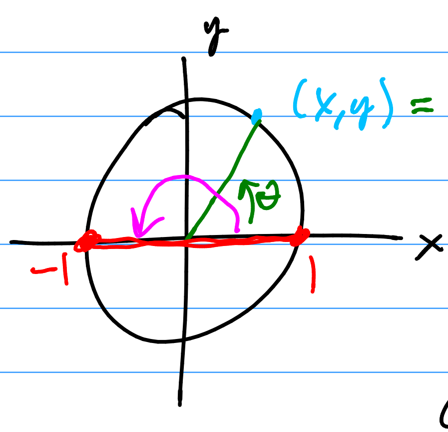
$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - 2\,\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$\|\vec{u}\|^2 - 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$-2\,\vec{u}\cdot\vec{v} = -2\,\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)}$$



$$\text{range}(\cos^{-1}) = [0, \pi]$$

$$\cos^{-1}(x) = \arccos(x)$$

Ex. $\|\vec{u}\| = 4$, $\|\vec{v}\| = 6$, $\theta = \pi/3$

Can we determine $\vec{u}\cdot\vec{v}$? Yes!

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$
$$= 4\cdot 6\cdot \cos(\pi/3) = 4\cdot 6\cdot \tfrac{1}{2} = 2\cdot 6 = 12$$

Ex. $\vec{u} = \langle 2, 2, -1\rangle$ $\vec{v} = \langle 5, -3, 2\rangle$

Find θ.

$\|\vec{u}\| = \sqrt{4+4+1} = \sqrt{9} = 3$

$\|\vec{v}\| = \sqrt{25+9+4} = \sqrt{38}$

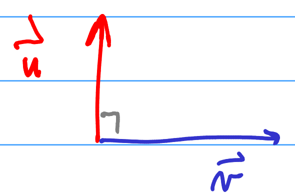
$\vec{u}\cdot\vec{v} = 10 - 6 - 2 = 2$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right) = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \text{ rad.}$$

Recall $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ for $\vec{u}, \vec{v} \neq \vec{0}$.

If $\vec{u}\cdot\vec{v} = 0$, what must be true of $\vec{u}, \vec{v}$?

then $\cos\theta = 0$ and $\theta = \pi/2 = 90^\circ$



Thm. If $\vec{u}, \vec{v} \neq \vec{0}$ and $\vec{u}\cdot\vec{v} = 0$, then $\vec{u} \perp \vec{v}$.

Def'n. Two vectors are said orthogonal if and only if their dot product is 0.

Ex. $\vec{u} = \langle -2, 1, 3\rangle$ $\vec{v} = \langle \tfrac{1}{2}, 1, 0\rangle$

Verify the Pythag. Theorem: $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\Longleftrightarrow$

$\vec{u}\cdot\vec{v} = 0$

$$\|\vec{u}\| = \sqrt{4+1+9} = \sqrt{14} \qquad \|\vec{v}\| = \sqrt{\tfrac{1}{4}+1+0} = \frac{\sqrt{5}}{2}$$

$$\|\vec{u}\|^2 = 14 \qquad \|\vec{v}\|^2 = 5/4$$

$$\vec{u}-\vec{v} = \langle -5/2,\ 0,\ 3\rangle$$

$$\left.\begin{aligned} \|\vec{u}-\vec{v}\|^2 &= (-5/2)^2 + 3^2 = \frac{25}{4} + \frac{36}{4} = \frac{61}{4} \\ \|\vec{u}\|^2 + \|\vec{v}\|^2 &= 14 + 5/4 = \frac{56}{4} + 5/4 = \frac{61}{4} \end{aligned}\right\} = \,!$$