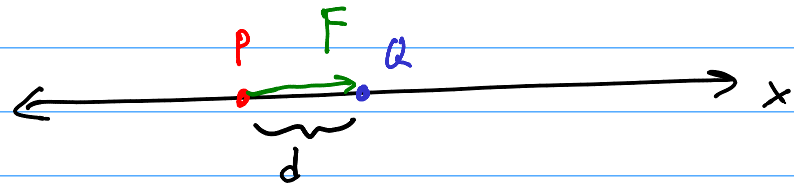
M243 — 10/1/18

§8.3 - Applications of Integration to Physics

Work: $W = Fd$



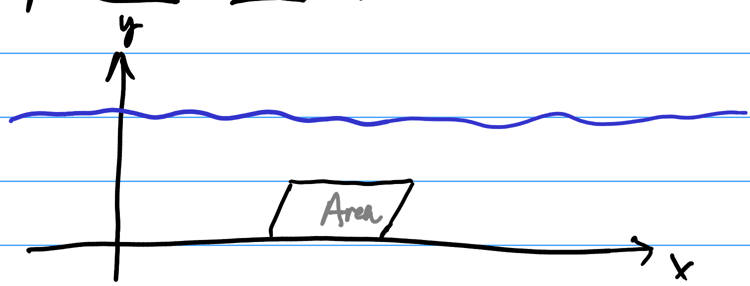
If F is a function of x, then $F(x)\,dx$ must be computed by an integral,

$$W = \int_a^b F(x)\,dx$$

Ex. P is located x units from the origin. A force of $F(x) = x^2 + 2x$ is exerted on P.
How much work is done in moving the particle from $x=1$ to $x=3$?

$$W = \int_1^3 x^2 + 2x \, dx = \frac{1}{3}x^3 + x^2 \Big|_1^3 = 9 + 9 - \frac{1}{3} - 1 = 17 - \frac{1}{3}$$
$$= \frac{50}{3} \text{ Nm}$$

Hydrostatic Force/Pressure:



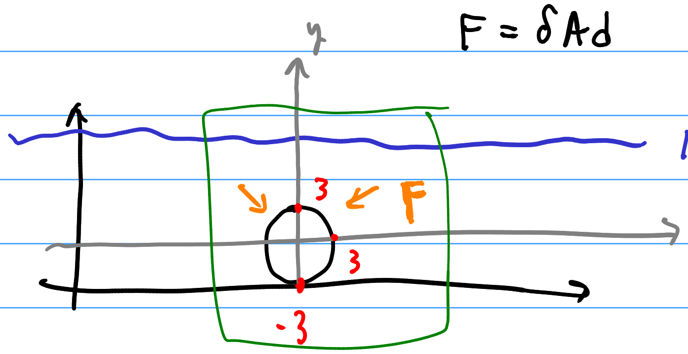
δ = weight density

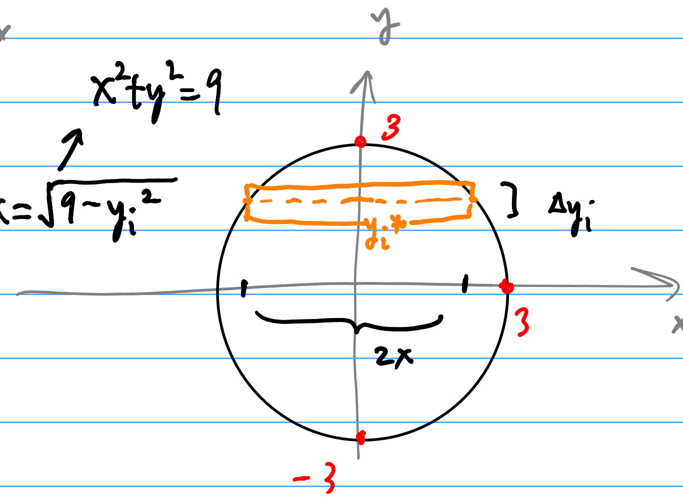
$$F_H = mg = \underbrace{\rho g}_{\delta} A d$$

ρ = density of fluid
g = acceleration due to grav.
A = area of the submerged object
d = depth

Ex. Cylindrical rod submerged in 10 ft of water.
rod has 3 ft. radius.
Compute the hydrostatic force on one face of the rod.

$$F = \delta A d$$





depth $= (7-y_i^*)$
$\delta = 62.5 \text{ lb/ft}^3$

$\Delta A_i = 2\sqrt{9-y_i^{*2}}\,\Delta y_i$

$$F = \lim_{N\to\infty} \sum_{i=1}^{N} 62.5\,(7-y_i^*)\cdot 2\sqrt{9-y_i^{*2}}\,\Delta y_i = \int_{-3}^{3} \underbrace{125}_{\delta}\,\underbrace{(7-y)}_{d}\,\underbrace{\sqrt{9-y^2}\,dy}_{dA}$$

$$F_H = 125\int_{-3}^{3} \underbrace{7\sqrt{9-y^2}}_{\text{EVEN}}\,dy - 125\int_{-3}^{3} \cancel{\overbrace{y\sqrt{9-y^2}}^{\text{ODD}}}_{0}\,dy$$

$$= 1750\int_0^3 \sqrt{9-y^2}\,dy$$

$y = 3\sin\theta$
$dy = 3\cos\theta\,d\theta$

$\theta = \sin^{-1}(y/3)$
$\theta(3) = \sin^{-1}(1) = \pi/2$
$\theta(0) = \sin^{-1}(0) = 0$

$$= 1750\int_0^{\pi/2} 3\cos\theta\,(3\cos\theta)\,d\theta$$

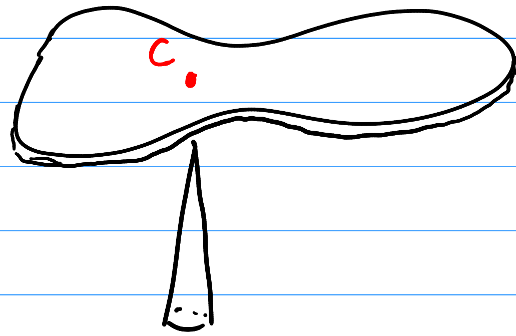
$$= 15750\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{15750}{2}\int_0^{\pi/2} 1+\cos(2\theta)\,d\theta$$

$$\frac{15{,}750}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/2} = \frac{15{,}750\,\pi}{4} \text{ lb.}$$

$$\approx 12{,}370 \text{ lb.}$$

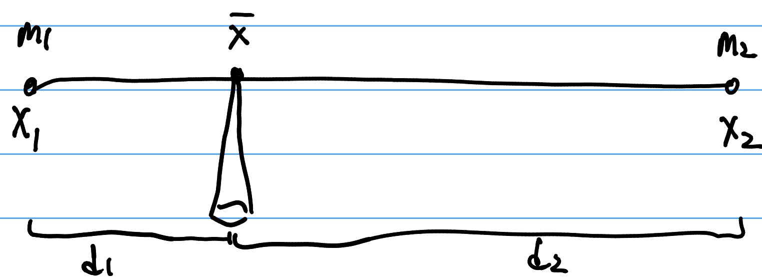
Balancing Fulcrum:

"lamina" = thin sheet



Goal: Find center of mass

First, consider a long, thin rod



Point masses m_1 at x_1 and m_2 at x_2.

Where is the center of mass? $\bar{x}$?

$d_1 = \bar{x} - x_1$

$d_2 = x_2 - \bar{x}$

Archimedes: $m_1 d_1 = m_2 d_2$

Plug in:

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \qquad \text{solve for } \bar{x}$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

In general, if we have N point-masses, then

$$\bar{x} = \sum_{k=1}^{N} \frac{x_k m_k}{m} \qquad \text{where } m = \sum_{k=1}^{N} m_k$$

$$\bar{x} = \frac{1}{m} \underbrace{\sum_{k=1}^{N} x_k m_k}$$

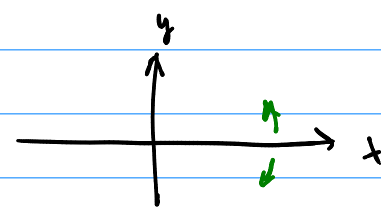
$$M = \sum_{k=1}^{N} x_k m_k$$ is the <u>moment</u>.

In a 2D system, $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$.

the moments are:

$$\begin{cases} M_y = \displaystyle\sum_{k=1}^{N} m_k x_k \\ M_x = \displaystyle\sum_{k=1}^{N} m_k y_k \end{cases}$$

M_y measures the tendency of the plate to rotate in the y-direction

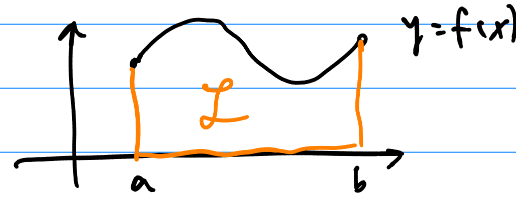


The center of mass is:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \qquad m = \sum_{k=1}^{N} m_k$$

For a lamina, the mass at each pt is given by a density function, $\rho(x,y) = \rho(x)$.

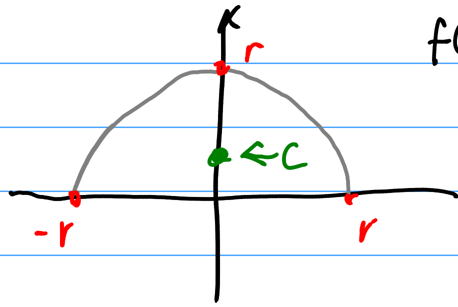
For a lamina bounded by $y = f(x)$



$$\boxed{\begin{aligned} M_y &= \rho \int_a^b x f(x)\, dx \\ M_x &= \rho \int_a^b \frac{1}{2} [f(x)]^2\, dx \end{aligned}}$$ for constant density ρ.

Ex. Find the center of mass of a semicircle $x^2+y^2=r^2$, $y \geq 0$.

$\rho = 1$.



$f(x) = \sqrt{r^2 - x^2}$

$a = -r$

$b = r$

$A = \frac{1}{2}\pi r^2$

Compute M_x, M_y, Area of shape.

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{A}, \frac{M_x}{A}\right)$$

$$M_y = \int_{-r}^{r} x\sqrt{r^2-x^2}\,dx = 0$$

$$M_x = \int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2-x^2}\right)^2 dx = \int_{0}^{r} r^2 - x^2\,dx = r^2x - \frac{1}{3}x^3\Big|_0^r$$

$$= r^3 - \frac{1}{3}r^3$$

$$M_x = \frac{2}{3}r^3$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(0, \frac{\frac{2}{3}r^3}{\frac{1}{2}\pi r^2}\right) = \boxed{\left(0, \frac{4r}{3\pi}\right) = C}$$