M243 1 Nov '18

Ch9 ~ Second Order Linear Homogeneous Constant Coefficient DEs.

$$ay'' + by' + cy = 0$$

solve for $y = y(x)$
a, b, c are constant
$a \neq 0$.

"Guess" a solution by first solving

$$\frac{by'}{b} + \frac{cy}{b} = 0 \qquad b \neq 0$$

$$y' = \frac{-c}{b} y$$

$$\frac{dy}{dx} = \underbrace{\boxed{-\frac{c}{b}}}_{r} y$$

$$\frac{dy}{dx} = ry \longrightarrow \int \frac{1}{y}\, dy = \int r\, dx$$

separable!

$$\ln y = rx + C$$

$$y = e^{rx + C}$$

$$\boxed{y = Ce^{rx}} \leftarrow \text{Guess this!}$$

So, $ay'' + by' + cy = 0$

$$\begin{cases} y = e^{rx} \\ y' = re^{rx} \\ y'' = r^2 e^{rx} \end{cases}$$

Q: what is r?

substituting in,

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$\underbrace{e^{rx}}_{\neq 0}\underbrace{(ar^2 + br + c)}_{=0} = 0$$

$$ar^2 + br + c = 0$$

$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = r_1, r_2$$

The general solution of $ay'' + by' + cy = 0$ is

$$\longrightarrow \boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

Ex. $\begin{cases} y'' + 5y' + 6y = 0 \leftarrow \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$

The characteristic eqn is

$$r^2 + 5r + 6 = 0$$
$$(r+3)(r+2) = 0$$
$$r = -3, \quad r = -2 \leftarrow$$

General solution, $y(x) = C_1 e^{-3x} + C_2 e^{-2x}$

So, $y'(x) = -3C_1 e^{-3x} - 2C_2 e^{-2x}$

$$\begin{cases} y(0) = C_1 + C_2 = 2 \\ y'(0) = -3C_1 - 2C_2 = 3 \end{cases}$$

Cramer! $A = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$ $\quad \bar{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$C_1 = \frac{|A_1|}{|A|} \qquad C_2 = \frac{|A_2|}{|A|}$$

$A_1 = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$

$A_2 = \begin{pmatrix} 1 & 2 \\ -3 & 3 \end{pmatrix}$

$|A| = -2 + 3 = 1$

$|A_1| = -4 - 3 = -7$

$|A_2| = 3 + 6 = 9$

$$\boxed{\begin{matrix} C_1 = -7 \\ C_2 = 9 \end{matrix}}$$

$$\boxed{y(x) = -7e^{-3x} + 9e^{-2x}}$$ Particular solution of IVP.

Two possibilities: if the characteristic polynomial has a real repeated root, $r = r_1$, then the general solution is

$$\boxed{y(x) = C_1 e^{rx} + C_2 x e^{rx}} \quad *$$

Check it: $ay'' + by' + cy = 0$

$y_2 = xe^{rx}$

$y_2' = (1+rx)e^{rx}$

$y_2'' = (r + r + r^2x)e^{rx}$

$= (2r + r^2x)e^{rx}$

$\cancel{e^{rx}}\left[a(2r+r^2x) + b(1+rx) + cX\right] \overset{?}{=} 0$

$\left[2ar + ar^2x + b + brx + cx\right]$

$= \left[2ar + b + x\underbrace{(ar^2+br+c)}_{=0}\right]$

$r = \frac{-b}{2a} \pm 0 = \frac{-b}{2a}$

$= 2ar + b$

$= \cancel{2a}\left(\frac{-b}{\cancel{2a}}\right) + b$

$= 0$ ✓

Euler's Equation: $e^{ix} = \cos x + i\sin x$

$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0$

$$\boxed{e^{i\pi} + 1 = 0}$$

$r = a \pm bi$, $y(x) = C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx)$

Ex. $\underbrace{y'' = -y}$ $\longrightarrow$ $y'' + 0y' + 1y = 0$

$y = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm\sqrt{-1} = \pm i = 0 \pm 1i$

$$\boxed{y(x) = C_1\cos x + C_2\sin x}$$ ✓