M243, Calculus II 15 Nov '18

§11.3–11.6 Exercises (which is all the content of §11.7)

1. Use the limit definition of series to find the exact sum of the convergent series:

   a.) $\sum_{n=1}^{\infty} \frac{6\cdot 3^n}{2^{2n-1}}$ b.) $10-2+0.4-0.08+\cdots$ c.) $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$

   or an equivalent formula/method, when appropriate.

2. Determine whether the series are convergent or divergent. Do not find the sum. Clearly state which convergence test(s) you apply.

TD
IT
pT
CT
LCT
AST
RaT
RoT

a) $\sum_{n=1}^{\infty}\frac{n^2-1}{n^3+1}$ d.) $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ g.) $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)^n$

b.) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ e.) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ h.) $\sum_{n=1}^{\infty}(-1)^n\frac{n^2-1}{n^2+1}$

c.) $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$ f.) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{3/2}}$ i.) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Brief Solutions

1. a.) $\sum_{n=1}^{\infty}\frac{6\cdot 3^n}{2^{2n-1}} = \sum_{n=1}^{\infty}\frac{6\cdot 3^n}{2^{-1}4^n} = \sum_{n=1}^{\infty} 12\left(\frac{3}{4}\right)^n = \frac{12}{1-3/4} - 12 = 48-12 = 36.$

b.) $10-2+0.4-0.08+\cdots = \sum_{n=0}^{\infty} 10\left(\frac{-1}{5}\right)^n = \frac{10}{1-(-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}$

c.) $\sum_{n=2}^{\infty}\frac{2}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$ by PFD.

$$S_k = \sum_{n=2}^{k}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\cdots+\left(\cancel{\frac{1}{k-2}}-\frac{1}{k}\right)+\left(\cancel{\frac{1}{k-1}}-\frac{1}{k+1}\right)$$

so, $S_k = 1+\frac{1}{2}-\left(\frac{1}{k}+\frac{1}{k+1}\right)$ and $S=\lim_{k\to\infty}S_k = 1+\frac{1}{2} = \frac{3}{2}.$

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2. a) $\sum_{n=1}^{\infty}\frac{n^2-1}{n^3+1}$ LCT w/ $\sum_{n=1}^{\infty}\frac{1}{n}$: $\frac{n^2-1}{n^3+1}\Big/\frac{1}{n} = \frac{n(n^2-1)}{n^3+1} \xrightarrow{n\to\infty} 1.$ So divergent since $\sum_{n=1}^{\infty}\frac{1}{n}$ is known to diverge.

b.) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ RaT: $\left| \frac{(n+1)^4}{4^{n+1}} \cdot \frac{4^n}{n^4} \right| = \frac{1}{4}\left(1+\frac{1}{n}\right)^4 \xrightarrow{n\to\infty} \frac{1}{4}(1)^4 = \frac{1}{4} < 1$

so, <u>absolutely convergent.</u>

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c.) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ AST: $b_n = \frac{1}{\sqrt{n}-1}$, $\frac{1}{\sqrt{n}-1} \xrightarrow{n\to\infty} 0$ ✓

$\frac{1}{\sqrt{n+1}-1} \boxed{<} \frac{1}{\sqrt{n}-1} \Rightarrow \sqrt{n} \cancel{-1} \boxed{<} \sqrt{n+1} \cancel{-1}$ ✓

<u>Convergent</u> by AST

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d.) $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ TD: $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \infty \cdot 0$ smells like L'Hôpital

$\lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{u\to 0^+} \frac{\sin u}{u} \overset{L'H}{=} \lim_{u\to 0^+} \frac{\cos u}{1} = \cos 0 = 1 \neq 0.$

Diverges!

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e.) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ $(\ln n)^{\ln n} = \left(e^{\ln(\ln n)}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln(\ln n)} = n^{\ln(\ln n)} > n^2$ for $n > e^e$.

Thus $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$ for $n > e^e$, and $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ <u>converges</u> by CT.

$\left(\sum_{n=2}^{\infty} \frac{1}{n^2} \text{ is convergent by pT.}\right)$

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f.) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ IT: $\int_2^{\infty} \frac{1}{(\ln x)^{3/2} x} dx \left\{ \begin{matrix} u = \ln x & u(\infty) = \infty \\ du = \frac{1}{x} dx & u(2) = \ln 2 \end{matrix} \right\} = \int_{\ln 2}^{\infty} \frac{1}{u^{3/2}} du$

<u>convergent</u> by pT. So $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ is Convergent by IT.

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g.) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)^n$ RoT: $\sqrt[n]{\left|\sqrt[n]{2}-1\right|^n} = \left|\sqrt[n]{2}-1\right| \longrightarrow 1-1=0$ Inconclusive

CT: $\sqrt[n]{2}-1 < \frac{1}{2}$ for all n, so $0 < \left(\sqrt[n]{2}-1\right)^n < \left(\frac{1}{2}\right)^n$

and $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent (geometric).

Hence $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)^n$ is <u>convergent</u>.

h.) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$

$\underline{TD}$: $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1 \neq 0$, so $\underline{\text{divergent}}$.

i.) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$\underline{II}$: $\int_1^{\infty} x^2 e^{-x^3} dx$ $u = x^3$ $u(\infty) = \infty$
$du = 3x^2 dx$ $u(1) = 1$

$= \int_1^{\infty} \frac{1}{3} e^{-u} du = \frac{-1}{3} e^{-u} \Big|_1^{\infty} = -\frac{1}{3} \cdot 0 + \left(\frac{1}{3} e^{-1}\right) = \frac{1}{3e}$ $\underline{\text{convergent}}$.