Calculus II - Series Exercises

Ex. Determine whether the series converge or diverge. Give adequate reasoning.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$
2. $\displaystyle\sum_{n=1}^{\infty} n^4 e^{-n^5}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{7^n n^2}{n!}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n-1} 3^{n+1}}{n^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{8}{n}\right)$
6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n+1)^n}{n^{8n}}$
8. $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n\sqrt{\ln(6n)}}$
9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^8} + \frac{1}{8^n}\right)$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(7n)}{1+7^n}$
11. $\displaystyle\sum_{n=1}^{\infty} \frac{7n!}{e^{n^2}}$
12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+5}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+4^n}$

Brief Solutions (sometimes very brief)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ TD: $\displaystyle\lim \frac{e^n}{n^2} \overset{L'H}{=} \lim \frac{e^n}{2n} = \lim \frac{e^n}{2} = \infty$ $\boxed{\text{Divergent}}$

2. $\displaystyle\sum_{n=1}^{\infty} n^4 e^{-n^5}$ IT: $\displaystyle\int_1^{\infty} x^4 e^{-x^5}\,dx \left\{\begin{matrix} u = x^5 \\ du = 5x^4 dx \end{matrix}\right\} = \frac{1}{5}\int_1^{\infty} e^{-u}\,du = -\frac{1}{5} e^{-u}\Big|_1^{\infty} = 0 + \frac{1}{5e} < \infty$ $\boxed{\text{Convergent}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{7^n n^2}{n!}$ RaT: $\displaystyle\left|\frac{7^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{7^n n^2}\right| = \frac{7\cancel{7^n}\cancel{(n+1)}(n+1)}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{7^n} n^2} = \frac{7(n+1)}{n^2} \xrightarrow{n\to\infty} 0$ $\boxed{\text{Convergent}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n-1} 3^{n+1}}{n^n}$ RaT: $\displaystyle\left|\frac{7^n 3^{n+2}}{(n+1)^{n+1}} \cdot \frac{n^n}{7^{n-1} 3^{n+1}}\right| = 21\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \xrightarrow{n\to\infty} \frac{21}{e}\cdot 0 = 0$ $\boxed{\text{Convergent}}$

5. $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{8}{n}\right)$ LCT: $\displaystyle\lim_{n\to\infty} \frac{\tan(8/n)}{8/n} = \lim_{u\to 0^+} \frac{\tan u}{u} \overset{L'H}{=} \lim_{0^+} \frac{\sec^2 u}{1} = (\sec 0)^2 = 1^2 = 1.$ $\boxed{\text{Divergent}}$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ RoT: $\displaystyle\lim \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$ $\boxed{\text{Convergent}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n+1)^n}{n^{8n}}$ RoT: $\displaystyle\lim \sqrt[n]{\frac{(3n+1)^n}{n^{8n}}} = \lim \frac{3n+1}{n^8} = 0 < 1$ $\boxed{\text{Convergent}}$

8. $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n\sqrt{\ln(6n)}}$ IT: $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{\ln(6x)}} \frac{1}{x}\,dx$ $\left\{\begin{matrix} u=\ln(6x) & u(1)=\ln 6 \\ du=\frac{1}{x}dx & u(\infty)=\infty \end{matrix}\right\}$ $= \displaystyle\int_{\ln 6}^{\infty} \frac{1}{\sqrt{u}}\,du = \infty$ by pT. $\boxed{\text{Divergent}}$

9. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^8}+\frac{1}{8^n}\right) = \overbrace{\sum_{n=1}^{\infty}\frac{1}{n^8}}^{*} + \overbrace{\sum_{n=1}^{\infty}\left(\frac{1}{8}\right)^n}^{**}$

$*$ converges by pT
$**$ converges as geometric series $\Big\}$ $\boxed{\text{Convergent}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(7n)}{1+7^n}$ Abs: $\displaystyle\left|\frac{\sin(7n)}{1+7^n}\right| = \frac{|\sin(7n)|}{1+7^n} \leq \frac{1}{1+7^n}$ for all n

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+7^n}$ converges by LCT: $\displaystyle\frac{7^n}{1+7^n} = \frac{1}{1+1/7^n} \xrightarrow{n\to\infty} 1$

So $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(7n)}{1+7^n}$ is $\boxed{\text{absolutely convergent}}$ by CT.

11. $\displaystyle\sum_{n=1}^{\infty}\frac{7n!}{e^{n^2}}$ RaT: $\displaystyle\left|\frac{7(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{7n!}\right| = \frac{\cancel{7}(n+1)\cancel{n!}}{\cancel{e^{n^2}}e^{2n}e}\cdot\frac{\cancel{e^{n^2}}}{\cancel{7}\cancel{n!}} = \frac{n+1}{e^{2n+1}} \xrightarrow{n\to\infty} 0$ $\boxed{\text{Convergent}}$

12. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+5}$ AST: $b_n = \dfrac{\sqrt{n}}{n+5} \xrightarrow{n\to\infty} 0$ ✓

$(b_n)' = \dfrac{(n+5)\frac{1}{2\sqrt{n}} - \sqrt{n}}{(n+5)^2} = \dfrac{n+5-2n}{2\sqrt{n}(n+5)^2} = \dfrac{-n+5}{2\sqrt{n}(n+5)^2} < 0$ for $n > 5$

So $b_{n+1} \leq b_n$ for $n > 5$ ✓ $\boxed{\text{Convergent}}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+4^n}$ CT: $\dfrac{1}{n+4^n} < \dfrac{1}{4^n} = \left(\dfrac{1}{4}\right)^n$ and $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ is convergent (geom. series)

So, $\boxed{\text{Convergent}}$