M243 27-Nov '18

§11.8 **Power Series** x_0 fixed number

$\{a_n\}$ sequence of numbers.

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots$$

Representing functions as P.S.

Ex. $f(x)=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ $|x|<1$

since it's geometric.

Ex. $f(x)=\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} 1\cdot(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ $\quad r=-x^2$ $\quad$ for

$a=1$ $\quad |x^2|<1$

$|r|<1$ $\quad |x|<1$

$$f(x)=\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for } -1<x<1.$$

Ex. $f(x)=\frac{1}{x+2}=\frac{1}{2+x}=\frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} = \frac{1/2}{1-(-\frac{x}{2})}$ $\quad a=1/2$, $r=-\frac{x}{2}$

$$\frac{1}{x+2}=\sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

$\left|\frac{x}{2}\right|<1$

$|x|<2$

$-2<x<2$

$\underline{\text{Ex}}$. $f(x) = \dfrac{x^3}{x+2} = \dfrac{\frac{1}{2}\boxed{x^3}}{1-\left(-\frac{x}{2}\right)}$ $\quad a = \frac{1}{2}x^3$ $\quad r = \left(-\frac{x}{2}\right)$

$$\frac{1}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \qquad \text{so}$$

$$\frac{x^3}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$

IC: still $(-2,2)$

$\underline{\text{Ex}}$. $f(x) = \dfrac{7x+1}{3x^2+2x-1} \overset{\text{PFD}}{=} \dfrac{A}{3x-1} + \dfrac{B}{x+1}$

$$3x^2+2x-1 = (3x^2+3x)+(-x-1) = 3x(x+1)-(x+1)$$
$$=(3x-1)(x+1)$$

$$7x+1 = A(x+1) + B(3x-1)$$

$x=-1$: $\quad -6 = -4B \Rightarrow B = 3/2$

$x=1/3$: $\quad 10/3 = 4/3\, A \Rightarrow A = 5/2$

$$\frac{7x+1}{3x^2+2x-1} = \frac{5}{2}\,\frac{1}{3x-1} + \frac{3}{2}\,\frac{1}{x+1}$$

$$= \frac{5}{2} \cdot \frac{-1}{1-3x} + \frac{3}{2} \cdot \frac{1}{1-(-x)}$$

$$= \sum_{n=0}^{\infty} \frac{-5}{2}(3x)^n + \sum_{n=0}^{\infty} \frac{3}{2}(-x)^n$$

$|3x|<1$ $\quad |x|<1$

$|x|<1/3$ $\qquad R = 1/3$

then,

$$\frac{7x+1}{3x^2+2x-1} = \sum_{n=0}^{\infty}\left[-\frac{5}{2}3^n + \frac{3}{2}(-1)^n\right]x^n \qquad \underline{R = 1/3}$$

Ex. $f(x) = \frac{3}{x^2+2x+2} \overset{\text{CtS}}{=} \frac{3}{(x^2+2x+1)-1+2} = \frac{3}{1+(x+1)^2}$

GEOM!

$$= \frac{\overset{=a}{3}}{1-\underbrace{\left(-(x+1)^2\right)}_{r}} = \sum_{n=0}^{\infty} 3\left(-(x+1)^2\right)^n$$

$$= \sum_{n=0}^{\infty} 3(-1)^n (x+1)^{2n}$$

$|r| < 1$

$|(x+1)^2| < 1$

$|x+1| < 1 \qquad \boxed{R = 1}$

$-1 < x+1 < 1$
$\;\;{-1} \qquad\; {-1} \qquad {-1}$

$$\boxed{\text{IC: } -2 < x < 0}$$

Ex. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

Can we use this to define a PS for $\arctan(x)$?

Idea: $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$, so we want to compute

$$\underline{\int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx}.$$

$$\int \sum_{n=0}^{\infty} (-1)^n \underbrace{x^{2n}}_{-1<x<1} dx = \int 1 - x^2 + x^4 - x^6 + x^8 - + \cdots + (-1)^n x^{2n} + \cdots \, dx$$

$$= C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + - \cdots \frac{(-1)^n}{2n+1} x^{2n+1} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \arctan(x)$$

Q. What should C be so that these are equal as functions?

Choose $x=0$: $\quad \underbrace{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} 0^{2n+1}}_{0} = \underbrace{\arctan(0)}_{=0}$

$$C + 0 = 0 \quad \text{so} \quad C = 0.$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

domain of arctan is $\mathbb{R}$, but not the series.

this series has domain $-1 < x \leq 1$

Test the endpoints: $\quad \arctan(-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (-1)^{2n+1} \quad$ C or D?

$$(-1)^{n+2n+1} = (-1)^{3n+1} = ?$$

$$\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cancel{1^{2n+1}} = \text{Conv.!}$$