

Name: Key
MA253: Calculus III (Spring 2017)
Instructor: Justin Ryan
Midterm Exam 1: Sections 12.1–12.3, 12*, 13.1–13.4



Read and follow all instructions.

Part I: True or False [1 point each]

Read each statement carefully. In the space provided, write **T** if the statement is always true, or **F** otherwise.

- F 1. If the point (a, b) is in the domain of the vector function $z = f(x, y)$, then $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$. Must be continuous.
- T 2. The partial derivative $\partial_x F(a, b)$ represents the slope of the tangent line to the graph of $z = F(x, y)$ at the point $(a, b, F(a, b))$ lying in the plane $y = b$.
- F 3. The unit binormal vector is given by $\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$. $\vec{B} = \vec{T} \times \vec{N}$, order matters
- T 4. The parabola $y = ax^2$ has curvature $\kappa = 2|a|$ at the origin.
- T 5. The acceleration $\mathbf{a}(t)$ of a particle moving along a space curve always lies in the osculating plane to the space curve.

Part II: Fill in the Blank [1 point each]

Choose the appropriate word or phrase from the word bank, and write its corresponding letter in the space provided.

Word Bank:

- | | | |
|-----------------|---------------|--------------|
| A. Acceleration | B. Velocity | C. Curvature |
| D. Arc Length | E. Torsion | F. Ellipses |
| G. Hyperbolas | H. Tangential | I. Circles |
| J. Normal | K. Binormal | L. Tangent |

- C 6. The _____ of a space curve at a point is defined by $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$.
- H 7. Let $v(t)$ denote the speed of a particle moving along a space curve. Then $\dot{v}(t)$ is the _____ component of the particle's acceleration.
- F 8. The level curves of the function $z = 4x^2 + y^2$ are _____.
- D 9. The _____ of a space curve is defined by $s(t) = \int_0^t \|\dot{\mathbf{r}}(u)\| du$.
- J 10. The unit _____ vector to a space curve at a point is given by $\frac{1}{\kappa} \frac{d\mathbf{T}}{ds}$.

Part III: Multiple Choice [5 points each]

Write the letter corresponding to the appropriate answer in the space provided.

11–14. Consider the vector function $\mathbf{r}(t) = \left\langle \frac{1}{1+t^2}, \sin(t), te^t \right\rangle$.

C 11. What is the domain of $\mathbf{r}$?

A. $(0, \infty)$

B. $(-\infty, 0)$

C. $(-\infty, \infty)$

D. $[0, \infty)$

A 12. Compute $\lim_{t \rightarrow 0} \mathbf{r}(t)$.

A. $\langle 1, 0, 0 \rangle$

B. $\langle 1, 1, 1 \rangle$

C. $\langle 0, 1, 1 \rangle$

D. $\langle 1, 1, 0 \rangle$

D 13. Compute $\dot{\mathbf{r}}(t)$.

A. $\left\langle \frac{-2t}{(1+t^2)^2}, -\cos t, e^t + te^t \right\rangle$

B. $\left\langle \frac{1}{2t}, \cos t, e^t \right\rangle$

C. $\left\langle \frac{-2t}{(1+t^2)^2}, \cos t, e^t \right\rangle$

D. $\left\langle \frac{-2t}{(1+t^2)^2}, \cos(t), (1+t)e^t \right\rangle$

A 14. Compute $\int \mathbf{r}(t) dt$.

A. $\langle \arctan(t), -\cos(t), (t-1)e^t \rangle + \mathbf{c}$

B. $\langle \arctan(t), \cos t, (t-1)e^t \rangle + \vec{c}$

C. $\langle \arctan(t), -\cos t, \frac{1}{2}t^2e^t \rangle + \vec{c}$

D. $\left\langle \frac{t}{t+\frac{1}{3}t^3}, -\cos t, \frac{1}{2}t^2e^t \right\rangle + \vec{c}$

15–17. Consider the vector function $\mathbf{r}(t) = \langle -t, -\sin t, \cos t \rangle$.

C 15. Find the arc length function $s(t)$ for $\mathbf{r}$ starting at $t = 0$ and moving in the positive t direction.

A. $s = \sqrt{2}$

B. $s = \sqrt{2t}$

C. $s = \sqrt{2}t$

D. $s = \frac{t}{\sqrt{2}}$

A 16. Reparametrize $\mathbf{r}$ with respect to arc length.

A. $\mathbf{r}(s) = \left\langle -\frac{s}{\sqrt{2}}, -\sin\left(\frac{s}{\sqrt{2}}\right), \cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle$

B. $\vec{r}(s) = \langle -s, -s \sin(s), \cos(s) \rangle$

C. $\vec{r}(s) = \langle -1, -\cos(s), -s \sin(s) \rangle$

D. $\vec{r}(s) = \left\langle -\frac{1}{\sqrt{2}}, -s \cdot \sin\left(\frac{t}{\sqrt{2}}\right), s \cdot \cos\left(\frac{t}{\sqrt{2}}\right) \right\rangle$

D 17. Compute the curvature function $\kappa(s)$.

A. $\kappa(s) = \frac{s}{\sqrt{2}}$

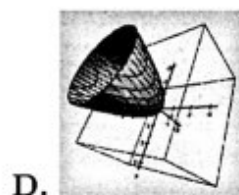
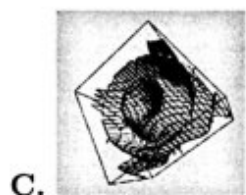
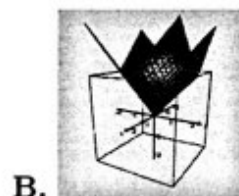
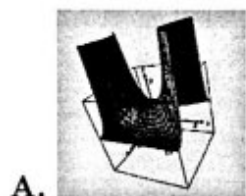
B. $\kappa(s) = s\sqrt{2}$

C. $\kappa(s) = \sqrt{2}$

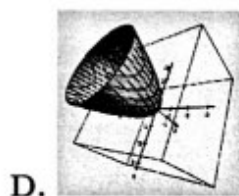
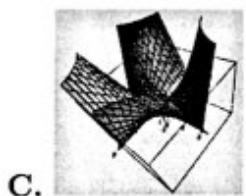
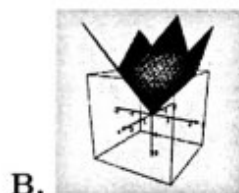
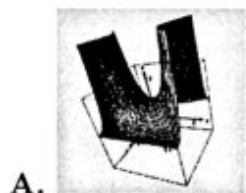
D. $\kappa(s) = \frac{1}{\sqrt{2}}$

18–20. Choose the graph that best corresponds to the function of two variables.

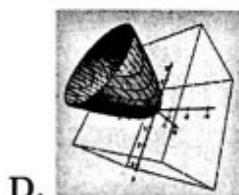
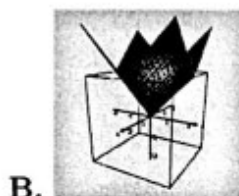
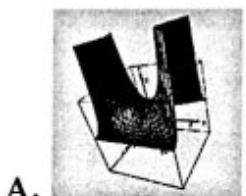
A 18. $f(x, y) = e^{-2x^2+y^2}$



B 19. $g(x, y) = |x| + |y|$



C 20. $F(x, y) = \sin(x^2 + y^2)$



21–22. Consider the function $F(x, y) = e^{xy} \sin y$.

A 21. Compute $\partial_x F$.

A. $e^{xy} y \sin y$

B. $e^{xy} y \cos y$

C. $e^{xy} x \sin y$

D. $e^y \sin y$

C 22. Compute $\partial_y F$.

A. $e^{xy} x \cos y$

B. $e^x \cos y$

C. $e^{xy} (\cos y + x \sin y)$

D. $e^y (\cos y - x \sin y)$

D 23. Compute $\frac{\partial^2 F}{\partial x \partial y}$ for the function $F(x, y) = x^y$.

A. $\ln(x) x^y$

B. $y x^{y-1}$

C. $\ln(x^y)(y-1)$

D. $y \ln(x) x^{y-1}$

D 24. Consider the relation $x^2 + y^2 + z^2 = xyz$. Find a formula for $\frac{\partial z}{\partial x}$.

A. $\frac{-2xyz}{2z + xy}$

B. $\frac{2x + yz}{2z + xy}$

C. $\frac{-2z + xy}{yz - 2x}$

D. $\frac{yz - 2x}{2z - xy}$

Part IV: Short Answer [5 points each]

You must complete four (4) of the problems in this part. Clearly write the word "OMIT" next to the problem that you wish not to be considered. Show enough work. Clearly mark your final answers. Partial credit given when deserved.

25. Consider the space curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ when $t = \pi$.

$$\dot{\mathbf{r}}(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\dot{\mathbf{r}}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} \\ = \sqrt{2}$$

$$s(t) = \int_0^t \sqrt{2} \, du = \sqrt{2}u \Big|_0^t = \sqrt{2}t$$

$$\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$

$$\vec{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds} = \left\langle -\frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right\rangle$$

$$\frac{d\vec{\mathbf{T}}}{ds} = \left\langle -\frac{1}{2} \cos\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{2} \sin\left(\frac{s}{\sqrt{2}}\right), 0 \right\rangle$$

$$\vec{\mathbf{N}}(s) = \left\langle -\cos\left(\frac{s}{\sqrt{2}}\right), -\sin\left(\frac{s}{\sqrt{2}}\right), 0 \right\rangle$$

$$\vec{\mathbf{T}}(\pi) = \left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{\mathbf{N}}(\pi) = \langle 1, 0, 0 \rangle$$

$$\vec{\mathbf{B}}(\pi) = \vec{\mathbf{T}}(\pi) \times \vec{\mathbf{N}}(\pi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{vmatrix}$$

$$\vec{\mathbf{B}}(\pi) = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

26. Show that the curvature of a circle of radius a is constant, $\kappa = \frac{1}{a}$.

$$\vec{r}(t) = \langle a \cos t, a \sin t, 0 \rangle$$

$$\dot{\vec{r}}(t) = \langle -a \sin t, a \cos t, 0 \rangle$$

$$\ddot{\vec{r}}(t) = \langle -a \cos t, -a \sin t, 0 \rangle$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 0, 0, a^2 \sin^2 t + a^2 \cos^2 t \rangle$$

$$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{(a^2)^2} = a^2$$

$$\|\dot{\vec{r}}\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a$$

$$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{a^2}{a^3} = \frac{1}{a}$$

So the curvature of the circle is constant, $\kappa(t) = \frac{1}{a}$. $\square$

27. Find an equation of the tangent plane to the curve $z = \sqrt{x^2 - y^2}$ at the point $(5, 3, 4)$.

$$\frac{\partial z}{\partial x} = \frac{2x}{2\sqrt{x^2 - y^2}} = \frac{x}{\sqrt{x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2\sqrt{x^2 - y^2}} = \frac{-y}{\sqrt{x^2 - y^2}}$$

$$z = z_1 + \frac{\partial z}{\partial x} (x - x_1) + \frac{\partial z}{\partial y} (y - y_1)$$

$$z = 4 + \frac{5}{4} (x - 5) - \frac{3}{4} (y - 3)$$

$$\left. \frac{\partial z}{\partial x} \right|_{(5, 3, 4)} = \frac{5}{4}$$

$$\left. \frac{\partial z}{\partial y} \right|_{(5, 3, 4)} = \frac{-3}{4}$$

28. Find an equation of the osculating circle to the curve $y = 4x^2 - x^4$ at the origin, $(0, 0)$.

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^3} = \frac{8}{\sqrt{1}^3} = 8.$$

$$f(x) = 4x^2 - x^4$$

$$f'(x) = 8x - 4x^3$$

$$f''(x) = 8 - 12x^2$$

$$f'(0) = 0$$

$$f''(0) = 8$$

The osculating circle is given by

$$x^2 + (y - \frac{1}{8})^2 = \frac{1}{64}$$

The radius of the circle is $r = \frac{1}{8}$.

Since $f''(0) > 0$, the curve is concave up.

The center is thus at $(0, \frac{1}{8})$.

29. Use either the linearization or differentials to approximate the value of $\sqrt{5.01^2 - 2.99^2}$. Leave your answer as a simplified fraction.

From problem 27., the linearization is given by

$$Lf(x, y) = 4 + \frac{5}{4}(x-5) - \frac{3}{4}(y-3)$$

or the differential is

$$df = \frac{5}{4}dx - \frac{3}{4}dy.$$

$$\text{Then } Lf(5.01, 2.99) = 4 + \frac{5}{4}(5.01-5) - \frac{3}{4}(2.99-3)$$

$$= 4 + \frac{5}{4}\left(\frac{1}{100}\right) - \frac{3}{4}\left(\frac{-1}{100}\right)$$

$$= 4 + \frac{8}{400}$$

$$= 4 + \frac{2}{100}$$

$$= \frac{402}{100} = 4.02$$

Thus,

$$\sqrt{5.01^2 - 2.99^2} \approx \frac{402}{100} = 4.02$$

$$\frac{201}{50}$$