1. Vector-Valued Functions

These Good Problems cover material from sections 12.1 – 12.3 of our book. Topics include an introduction to vector functions and their space curves; derivatives and integrals of vector functions; and arc length and curvature of space curves.

1. Consider the vector function $\mathbf{r}(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \rangle$.

a.) What is the domain of $\mathbf{r}$?

$\sqrt{4-t^2}$: $4-t^2 \geq 0 \Rightarrow t^2 \leq 4 \Rightarrow |t| \leq 2 \Rightarrow -2 \leq t \leq 2.$

e^{-3t} : $\mathbb{R}$

$\ln(t+1)$: $t+1 > 0 \Rightarrow t > -1$

so, $-1 < t \leq 2$

b.) Evaluate $\lim_{t\to 0} \mathbf{r}(t)$.

$$\lim_{t\to 0} \vec{r}(t) = \left\langle \lim_{t\to 0} \sqrt{4-t^2}, \lim_{t\to 0} e^{-3t}, \lim_{t\to 0} \ln(t+1) \right\rangle = \left\langle \sqrt{4-0^2}, e^0, \ln(0+1) \right\rangle$$

$$= \langle 2, 1, 0 \rangle$$

2. Evaluate the limit.

$$\lim_{t\to 1} \left(\frac{t^2-t}{t-1}\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin(\pi t)}{\ln t}\mathbf{k} \right)$$

$$= \lim_{t\to 1} \left(\frac{t^2-t}{t-1} \right) \vec{i} + \lim_{t\to 1} \sqrt{t+8}\,\vec{j} + \lim_{t\to 1} \frac{\sin(\pi t)}{\ln t}\vec{k}$$

$$= \lim_{t\to 1} \left(\frac{t\,\cancel{(t-1)}}{\cancel{t-1}} \right) \vec{i} + \sqrt{1+8}\,\vec{j} + \lim_{t\to 1} \frac{\pi \cos(\pi t)}{1/t}\vec{j}$$

$$= 1\vec{i} + 3\vec{j} + (-\pi)\vec{k}$$

$$= \langle 1, 3, -\pi \rangle$$

3. Recall that a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is said to be *continuous* at the point $t = a$ if and only if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

We proved in class that if the component functions x, y, and z are each continuous at $t = a$, then $\mathbf{r}$ is continuous at $t = a$. Prove the converse: If $\mathbf{r}$ is continuous at $t = a$, then so are each of x, y, and z.

Proof: Let $\vec{r}$ be continuous at $t=a$.

On one hand, $\vec{r}(a) = \langle x(a), y(a), z(a)\rangle$ is defined.

On the other hand,

$\vec{r}(a) = \lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\rangle$.

Putting these together, $\langle x(a), y(a), z(a)\rangle = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\rangle$

and each of x, y, z are continuous. $\square$

4. Find a parametrization of the space curve defined by the intersection of the surfaces $z = 4x^2 + y^2$ and $y = x^2$ in $\mathbb{R}^3$.

Let $x = t$,

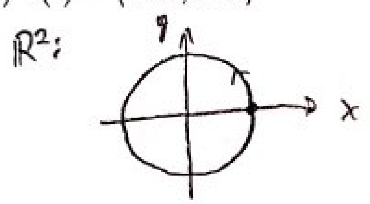
Then $y = t^2$ and $z = 4t^2 + (t^2)^2$.

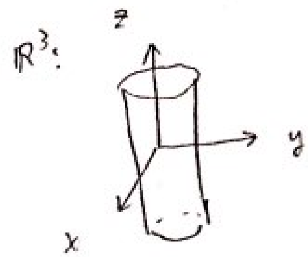
Thus,

$\vec{r}(t) = \langle t, t^2, 4t^2 + t^4\rangle$

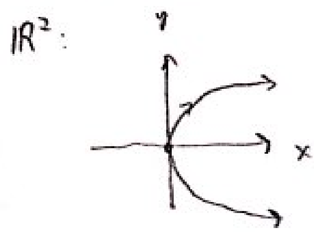
5. Sketch the curves in $\mathbb{R}^2$ and the surfaces $\mathbb{R}^3$ defined by the vector functions. Indicate the direction of increasing t.

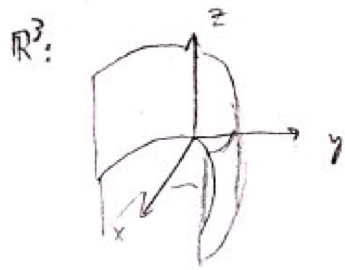
a.) $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$





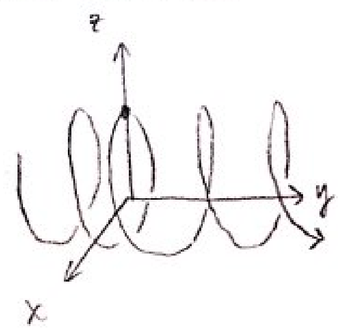
b.) $\mathbf{r}(t) = \langle t^2, t \rangle$





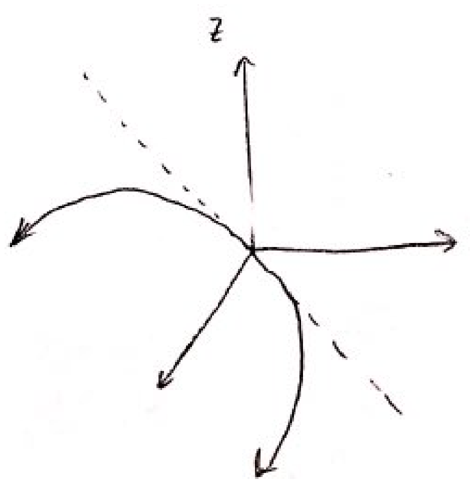
6. Sketch the space curves determined by the vector functions.

a.) $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$



"helix"

b.) $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} - t\,\mathbf{k}$



Bezier curves.

Let P_0, P_1, P_2, and P_3 be points in $\mathbb{R}^3$: $P_i = (x_i, y_i, z_i)$ for $i = 0, 1, 2, 3$. Regard each point P_i as the terminal point of a vector $\mathbf{P}_i$, identifying the ordered triple (x_i, y_i, z_i) with the vector $\langle x_i, y_i, z_i \rangle$ in $\mathbb{R}^3$. The *Bezier curve* defined by these points (equivalently, vectors) is the space curve associated with the vector function,

$$\mathbf{B}(t) = (1-t)^3\,\mathbf{P}_0 + 3(1-t)^2 t\,\mathbf{P}_1 + 3(1-t)t^2\,\mathbf{P}_2 + t^3\,\mathbf{P}_3, \quad 0 \le t \le 1.$$

7. Determine the Bezier curve for the points

$$P_0(0,1,2),\ P_1(0,0,2),\ P_2(2,1,0),\ \text{and } P_3(2,0,0).$$

Use a graphing utility to graph the curve. Identify each of the points and its relationship to the graph.

$$\vec{B}(t) = (1-t)^3\langle 0,1,2\rangle + 3(1-t)^2 t\langle 0,0,2\rangle + 3(1-t)t^2\langle 2,1,0\rangle + t^3\langle 2,0,0\rangle$$

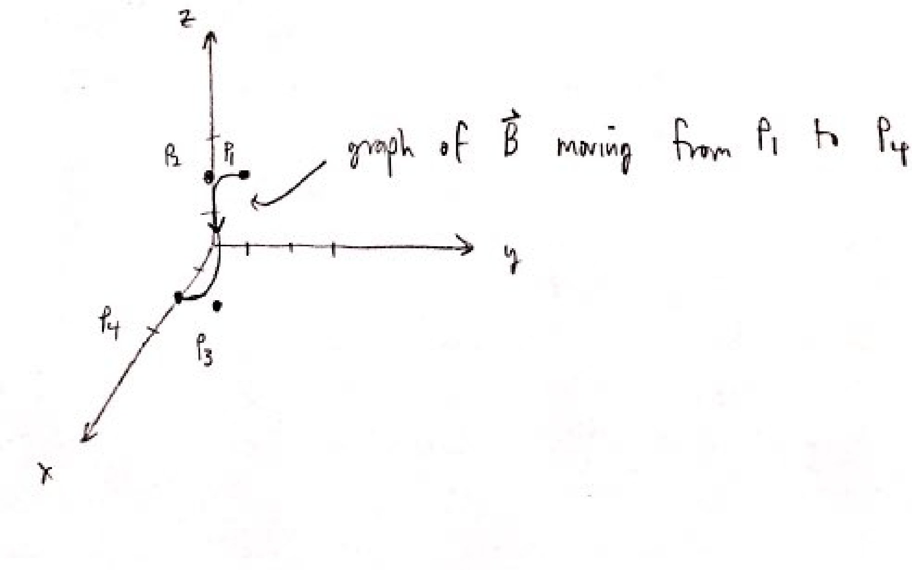
$$= \langle 0,(1-t)^3, 2(1-t)^3\rangle + \langle 0,0,6(1-t)^2 t\rangle + \langle 6(1-t)t^2, 3(1-t)t^2, 0\rangle + \langle 2t^3, 0, 0\rangle$$

$$= \langle 6(1-t)t^2 + 2t^3,\ (1-t)^3 + 3(1-t)t^2,\ 2(1-t)^3 + 6(1-t)^2 t\rangle$$

$$= \langle 2t^2(3(1-t)+t),\ (1-t)((1-t)^2+3t^2),\ 2(1-t)^2((1-t)+3t)\rangle$$

$$= \langle 2t^2(3-2t),\ (1-t)(1-2t+4t^2),\ 2(1-2t+t^2)(1+2t)\rangle$$

$$\boxed{\vec{B}(t) = \langle 6t^2-4t^3,\ 1-3t+6t^2-4t^3,\ 2-6t^2+4t^3\rangle}$$



8. Set up and simplify the integral that represents the length of the Bezier curve you found in problem 7. Use a graphing utility or CAS to estimate the length of the curve. Round your answer to two decimal places.

Recall the arc length is given by $s = \int_0^1 \|\dot{\vec{r}}(t)\| \, dt$.

$$\boxed{s = \int_0^1 3\sqrt{1 - 8t + 56t^2 - 96t^3 + 48t^4} \; dt}$$

$\dot{x}(t) = 12t - 12t^2$

$\dot{y}(t) = -3 + 12t - 12t^2$

$\dot{z}(t) = -12t + 12t^2$

Using WolframAlpha to compute:

$$\boxed{s \approx 3.31}$$

9. Let $y = f(x)$ be a twice-differentiable function. Show that the curvature of f is given by

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}^3}.$$

$\vec{r}(x) = \langle x, f(x), 0 \rangle$

$\dot{\vec{r}}(x) = \langle 1, f'(x), 0 \rangle$

$\ddot{\vec{r}}(x) = \langle 0, f''(x), 0 \rangle$

$\|\dot{\vec{r}}\| = \sqrt{1 + f'(x)^2}$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \langle 0, 0, f''(x) \rangle$$

So, $\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = |f''(x)|$

Thus,

$$\boxed{\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + f'(x)^2}\right)^3}}$$

10. Find a formula for the curvature of the curve $y = \tan x$, and use it to calculate the curvature at the point $(\frac{\pi}{4}, 1)$.

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$

So,

$$\boxed{\kappa(x) = \frac{2\sec^2 x\,|\tan x|}{\left(\sqrt{1+\sec^4 x}\right)^3}}$$

$\tan \pi/4 = 1$

$\sec \pi/4 = \sqrt{2}$

$$\Rightarrow \kappa(\pi/4) = \frac{2(\sqrt{2})^2(1)}{\sqrt{1+(\sqrt{2})^4}^{\,3}} = \frac{4}{\sqrt{5}^3}$$

11. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad \text{and} \quad \mathbf{r}_2(u) = \langle 1+2u, 1+6u, 1+14u \rangle.$$

Do the particles collide? If not, do their paths intersect?

$t = 1+2u \Rightarrow u = \frac{1}{2}(t-1)$

$t^2 = 1+6u \Rightarrow t^2 = 1+6(\frac{1}{2})(t-1) \Rightarrow t^2 = 1+3t-3 \Rightarrow t^2-3t+2=0$

$(t-2)(t-1)=0$

$t=1,\ t=2$

$t^3 = 1+14u$

$t^3 = 1+14(\frac{1}{2})(t-1)$

$t^3 = 1+7t-7$

$t^3-7t+6=0$

$t^3-t-6t+6=0$

$t(t+1)(t-1)-6(t-1)=0$

$(t-1)(t^2+t-6)=0$

$(t-1)(t-2)(t+3)=0$

$t=1,2,\ -3$

The paths intersect at $t=1$ and $t=2$.
when $t=1$, $u=0$, so the particles do not collide.
when $t=2$, $u=\frac{1}{2}$, so again the particles do not collide.

12. Consider the vector function $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$, $0 \le t \le 2\pi$. Compute $\dot{\mathbf{r}}(t)$, $\int \mathbf{r}(t)\,dt$, $s(t)$, and $\kappa(t)$.

$$\boxed{\dot{\vec{r}}(t) = \langle -3\sin t, 3\cos t, 1\rangle}$$

$$\boxed{\int \vec{r}(t)dt = \langle 3\sin t, -3\cos t, \tfrac{1}{2}t^2\rangle + \vec{C}}$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 1^2}$$

$$= \sqrt{9+1}$$

$$= \sqrt{10}$$

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\|du = \int_0^t \sqrt{10}\,du = u\sqrt{10}\Big|_0^t = t\sqrt{10}$$

$$\boxed{s(t) = t\sqrt{10}}$$

$$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin t & 3\cos t & 1 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} = (3\sin t)\vec{i} - (-3\cos t)\vec{j} + (9\sin^2 t + 9\cos^2 t)\vec{k}$$

$$= \langle 3\sin t, 3\cos t, 9\rangle$$

$$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{9\sin^2 t + 9\cos^2 t + 81} = \sqrt{90} = 3\sqrt{10}$$

$$\boxed{\kappa(t) = \frac{3\sqrt{10}}{10\sqrt{10}} = \frac{3}{10}}$$

13. Find an equation of a parabola that has curvature $\kappa = 4$ at the origin.

General parabola : $f(x) = a(x-h)^2 + k$

w/ origin as vertex: $f(x) = ax^2$

$f'(x) = 2ax$

$f''(x) = 2a$

$$\kappa(0) = \frac{|f''(0)|}{\sqrt{1+f'(0)^2}^3} = \frac{2a}{\sqrt{1+0^2}^3} = 2a = 4 \Rightarrow a = 2$$

Thus, $\boxed{f(x) = 2x^2}$

14. Use your favorite formula for curvature to prove the following statement: *The curvature of a circle of radius a is constant,* $\kappa = \dfrac{1}{a}$.

$\vec{r}(t) = \langle a\cos t, a\sin t, 0\rangle$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t, 0\rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin t + a^2\cos t} = \sqrt{a^2} = a.$

$s = \int_0^t a\,du = at \rightsquigarrow t = \frac{s}{a}$

$\vec{r}(s) = \langle a\cos(\frac{s}{a}), a\sin(\frac{s}{a}), 0\rangle$

$\frac{d\vec{r}}{ds} = \langle -\sin(\frac{s}{a}), \cos(\frac{s}{a}), 0\rangle = \vec{T}$

Since $\|\frac{d\vec{r}}{ds}\| = 1.$

$\frac{d\vec{T}}{ds} = \langle -\frac{1}{a}\cos(\frac{s}{a}), \frac{1}{a}\sin(\frac{s}{a}), 0\rangle$

$\|\frac{d\vec{T}}{ds}\| = \sqrt{\frac{1}{a^2}\cos^2(\frac{s}{a}) + \frac{1}{a^2}\sin(\frac{s}{a})}$

$= \sqrt{\frac{1}{a^2}}$

$= \frac{1}{a}$

hence,

$\kappa(s) = \|\frac{d\vec{T}}{ds}\| = \frac{1}{a}$ $\square$

For this reason, the number $1/\kappa$ is referred to as the *radius of curvature* at each point of a space curve.