4. Partial Derivatives, Tangent Planes, Review

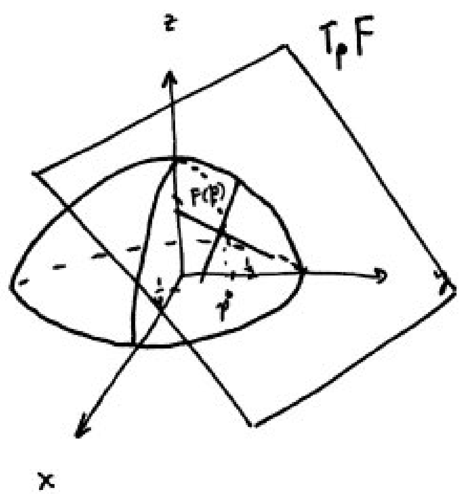
These Good Problems cover concepts studied in sections 13.3 and 13.4 of our text, then review concepts from the entire semester as preparation for the semester's first midterm exam.

1. *a.*) Sketch the graph of the function $F(x, y) = \sqrt{1 - x^2 - y^2}$.

b.) Plot the point $p(\frac{1}{4}, \frac{1}{2})$, in the domain of F, and the point $(\frac{1}{4}, \frac{1}{2}, F(p))$ on the surface.

c.) Sketch the tangent lines to the surface in the planes $x = \frac{1}{4}$ and $y = \frac{1}{2}$. Describe the partial derivatives $\partial_y F(p)$ and $\partial_x F(p)$ in terms of these lines.

d.) Sketch the tangent plane to the surface at the point $(\frac{1}{4}, \frac{1}{2}, F(p))$.



2. Calculate the partial derivatives $\partial_x F$ and $\partial_y F$ for the function $F(x,y) = \sqrt{1-x^2-y^2}$. Then use them to find an equation of the tangent plane to the surface at the point $(\frac{1}{4}, \frac{1}{2})$.

$$\partial_x F = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2-y^2}} = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\partial_y F = \frac{-y}{\sqrt{1-x^2-y^2}}$$ by symmetry (be carefull!)

$$F(\tfrac{1}{4},\tfrac{1}{2}) = \sqrt{1-\tfrac{1}{16}-\tfrac{1}{4}} = \sqrt{\tfrac{16-1-4}{16}} = \frac{\sqrt{11}}{4}$$

$$\partial_x F(\tfrac{1}{4},\tfrac{1}{2}) = \frac{-1/4}{\sqrt{11}/4} = \frac{-1}{\sqrt{11}}$$

$$\partial_y F(\tfrac{1}{4},\tfrac{1}{2}) = \frac{-1/2}{\sqrt{11}/4} = \frac{-2}{\sqrt{11}}$$

So, the tangent plane is: $\boxed{z = \frac{\sqrt{11}}{4} - \frac{1}{\sqrt{11}}(x-\tfrac{1}{4}) - \frac{2}{\sqrt{11}}(y-\tfrac{1}{2})}$

3. Compute the partial derivatives $\partial_x z$ and $\partial_y z$ for the function $z = \sin(xy) + ye^x$.

$$\partial_x z = y\cos(xy) + ye^x$$

$$\partial_y z = x\cos(xy) + e^x$$

4. Find an equation of the tangent plane to the surface

$$x^4 + y^4 + z^4 = 3x^2y^2z^2$$

at the point $(1, 1, 1)$.

Apply $\partial/\partial x$ to the entire equation (implicit differentiation):

$$\frac{\partial}{\partial x}\left[x^4 + y^4 + z^4 = 3x^2y^2z^2\right] \to 4x^3 + 4z^3\frac{\partial z}{\partial x} = 6xy^2z^2 + 6x^2y^2z\frac{\partial z}{\partial x}$$

$$\to 4z^3\frac{\partial z}{\partial x} - 6x^2y^2z\frac{\partial z}{\partial x} = 6xy^2z^2 - 4x^3$$

$$\to \frac{\partial z}{\partial x} = \frac{6xy^2z^2 - 4x^3}{4z^3 - 6x^2y^2z}$$

By symmetry, $\frac{\partial z}{\partial y} = \frac{6yx^2z^2 - 4y^3}{4z^3 - 6x^2y^2z}$

At the point $(1,1,1)$, we obtain $\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \frac{6-4}{4-6} = -1$ and $\frac{\partial z}{\partial y} = -1$.

So, the tangent plane is given by $\boxed{z = 1 - 1(x-1) - 1(y-1)}$

5. Find the differential dT of the function $T(u, v, w) = \dfrac{v}{1 + uvw}$.

$$\partial_u T = \frac{-v^2w}{(1+uvw)^2}$$

$$\partial_v T = \frac{(1+uvw) - uvw}{(1+uvw)^2} = \frac{1}{(1+uvw)^2}$$

$$\partial_w T = \frac{-v^2u}{(1+uvw)^2}$$

The differential $dT = \partial_u T\, du + \partial_v T\, dv + \partial_w T\, dw$

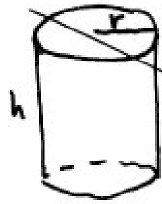
$$\boxed{dT = \frac{-v^2w}{(1+uvw)^2}du + \frac{1}{(1+uvw)^2}dv + \frac{-v^2u}{(1+uvw)^2}dw}$$

This is wrong. !! ☹

See the addendum at the end of these solutions.

(I should have used volume instead of surface area.)

6. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.



Surface area is $A(r,h)=2\pi r^2+2\pi rh$

We want to know dA when $r=4$, $h=12$ and $dr=dh=0.04$.

$$dA=\partial_r A\,dr+\partial_h A\,dh$$

$$\partial_r A = 4\pi r+2\pi h$$

$$\partial_r A(4,12)=16\pi+24\pi=40\pi$$

$$\partial_h A = 2\pi r$$

$$\partial_h A(4,12)=8\pi$$

$$dA(4,12,0.4,0.4)=40\pi(0.04)+8\pi(0.04)$$

$$=\frac{40}{25}\pi+\frac{8}{25}\pi$$

$$=\frac{48\pi}{25}$$

$$\approx\boxed{6.03 \text{ cm}^2 \text{ of tin}}$$

7. Use differentials to approximate the value of f at the point $(5.01, 4.02)$.

$f(x,y)=\sqrt{x^2-y^2}$

$$\partial_x f=\frac{x}{\sqrt{x^2-y^2}}$$

$$\partial_y f=\frac{-y}{\sqrt{x^2-y^2}}$$

$$f(5,4)=\sqrt{25-16}=\sqrt{9}=3.$$

$$\partial_x f(5,4)=\frac{5}{3}$$

$$\partial_y f(5,4)=\frac{-4}{3}$$

$$dx=0.01=\frac{1}{100}$$

$$dy=0.02=\frac{2}{100}$$

$$L_p f(x,y)=3+\frac{5}{3}(x-5)-\frac{4}{3}(y-4)$$

or $f(x,y)\approx 3+\frac{5}{3}dx-\frac{4}{3}dy$

$$L_p f(5.01,4.02)=3+\frac{5}{3}(5.01-5)-\frac{4}{3}(4.02-4)$$

$$=3+\frac{5}{3}\cdot\frac{1}{100}-\frac{4}{3}\cdot\frac{2}{100}$$

$$=3+\frac{5-8}{300}$$

$$=3-\frac{1}{100}$$

$$=2.99$$

So, $\boxed{\sqrt{5.01^2-4.02^2}\approx 2.99}$

8. Compute all first partial derivatives of the functions. Show enough work.

a.) $f(x,y) = \dfrac{1}{\sqrt{9-x^2-y^2}}$

$\partial_x f = \dfrac{-x}{\sqrt{9-x^2-y^2}}$, $\partial_y f = \dfrac{-y}{\sqrt{9-x^2-y^2}}$

b.) $g(u,v) = ue^{\sin(uv)}$

$\partial_u g = e^{\sin(uv)} + uv\cos(uv)\,e^{\sin(uv)}$,

$\partial_v g = u^2\cos(uv)e^{\sin(uv)}$

c.) $F(x,y) = \ln(x^2 + \arctan y)$

$\partial_x F = \dfrac{2x}{x^2+\arctan y}$, $\partial_y F = \dfrac{1}{1+y^2} \cdot \dfrac{1}{x^2+\arctan y}$

d.) $z = \arcsin(x+y)$

$\partial_x z = \dfrac{1}{\sqrt{1-(x+y)^2}}$, $\partial_y z = \dfrac{1}{\sqrt{1-(x+y)^2}}$

e.) $T(x,y) = e^{-x^2-y^2}$

$\partial_x T = -2x\,e^{-x^2-y^2}$, $\partial_y T = -2y\,e^{-x^2-y^2}$

f.) $z = y^5\sin(\ln x)$

$\partial_y z = 5y^4\sin(\ln x)$,

$\partial_x z = \dfrac{y^5}{x}\cos(\ln x)$

9. Let $\mathbf{r}$ be a smooth vector function in $\mathbb{R}^3$ such that $\ddot{\mathbf{r}}$ exists and $\ddot{\mathbf{r}} \neq 0$. Show that $\dot{\mathbf{T}}(t) \perp \mathbf{T}(t)$ for all values of t in the domain of $\mathbf{r}$.

Recall that $\|\vec{T}(t)\| = 1$ for all t, and $\|\vec{T}(t)\|^2 = \vec{T}(t)\cdot\vec{T}(t)$.

Then $\frac{d}{dt}\left(\vec{T}(t)\cdot\vec{T}(t) = 1\right) \Rightarrow \dot{\vec{T}}(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\dot{\vec{T}}(t) = 0$

$\Rightarrow 2\dot{\vec{T}}(t)\cdot\vec{T}(t) = 0 \Rightarrow \dot{\vec{T}}(t)\cdot\vec{T}(t) = 0$

$\Rightarrow \dot{\vec{T}}(t) \perp \vec{T}(t)$. $\square$

10. Prove that the curvature of a circle of radius a is constant, $\kappa = \frac{1}{a}$.

Parametrize the circle by $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$.

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t\rangle$, $\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$s(t) = \int_0^t a\,du = at$, so $\vec{r}(s) = \left\langle a\cos\left(\frac{s}{a}\right), a\sin\left(\frac{s}{a}\right)\right\rangle$.

Then $\vec{T}(s) = \frac{d\vec{r}}{ds} = \left\langle -\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)\right\rangle$

$\frac{d\vec{T}}{ds} = \left\langle -\frac{1}{a}\cos\left(\frac{s}{a}\right), -\frac{1}{a}\sin\left(\frac{s}{a}\right)\right\rangle$, and $\kappa(s) = \left\|\frac{d\vec{T}}{ds}\right\| = \sqrt{\left(\frac{1}{a}\right)^2\cos^2\left(\frac{s}{a}\right) + \left(\frac{1}{a}\right)^2\sin^2\left(\frac{s}{a}\right)}$

$= \sqrt{\frac{1}{a^2}} = \frac{1}{a}$. $\square$

[This uses the definition of κ. (as opposed to the "other" formulas.)]

11. Let $\mathbf{r}$ be a smooth space curve such that $\ddot{\mathbf{r}}$ exists and $\ddot{\mathbf{r}} \neq 0$. Prove that $\mathbf{B}(t)$ is a unit vector for all t in the domain of $\mathbf{r}$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ by definition.

Also, $\|\vec{T}(t)\| = 1$ and $\|\vec{N}(t)\| = 1$, and the angle between $\vec{T}$ and $\vec{N}$ is $\frac{\pi}{2}$ $\left(\vec{N}(s) = \kappa(s)\,\frac{d\vec{T}}{ds}\right)$.

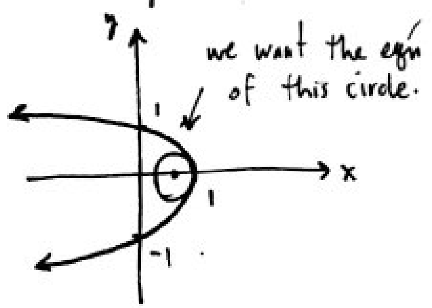
Thus,

$$\begin{aligned}\|\vec{B}(t)\| = \|\vec{T}(t)\times\vec{N}(t)\| &= \|\vec{T}(t)\|\,\|\vec{N}(t)\|\sin\theta \\ &= 1\cdot 1\cdot \sin(\pi/2) \\ &= 1\cdot 1\cdot 1 \\ &= 1.\end{aligned}$$

Thus, $\vec{B}(t)$ is a <u>unit</u> vector field. $\square$

12. Find an equation of the osculating circle to the plan curve $x = 1 - y^2$ at the point $(1, 0)$. Show enough work.

Center of the circle lies on $\vec{N}$.
radius is $\frac{1}{\kappa}$.
The circle should be tangent to the curve at the point.



Parametrize the curve
$y = t$, $x = 1 - t^2$
$\vec{r}(t) = \langle 1-t^2, t\rangle$
$\dot{\vec{r}}(t) = \langle -2t, 1\rangle$
$\ddot{\vec{r}}(t) = \langle -2, 0\rangle$
$\kappa(t) = \dfrac{\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|^3}$

$= \dfrac{2}{\sqrt{1+4t^2}^{\,3}}$

$\kappa(0) = \frac{2}{1} = 2.$

So $r = \frac{1}{2}$

The equation of the osculating circle is then $\boxed{(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}}$.

13. At what point(s) does the curve $y = x^4 - 6x^2$ have maximum curvature? Show enough work to justify your answer.

$f(x) = x^4 - 6x^2$

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

$\kappa(x) = \dfrac{|12x^2-12|}{\sqrt{1+(4x^3-12x)^2}^{\,3}}$

$$\kappa'(x) = \frac{\sqrt{1+(4x^3-12x)^2}^{\,3}(24x) - (12x^2-12)\frac{3}{\cancel{2}}\left(1+(4x^3-12x)^2\right)^{1/2}\cancel{2}(4x^3-12x)(12x^2-12)}{\left(1+(4x^3-12x)\right)^3}$$

$$= \frac{\sqrt{1+(4x^3-12x)^2}\left(24x - (12x^2-12)^2 (4x^2-12x)\right)}{\left(1+(4x^3-12x)\right)^3}$$

This can only equal 0 if $(12x^2-12)^2(4x^3-12x) - 24x = 0$

Real solutions are $x=0$ and two <u>really</u> obscene numbers. (I won't put something anywhere near this crazy on the exam.) Therefore, as far as we're concerned, $y = x^4 - 6x^2$ has maximum curvature at the point $(0,0)$.

14. Consider the space curve $\mathbf{r}(t) = \langle -\cos t, -t, \sin t \rangle$.

a.) Find the arc length function $s = s(t)$ starting at $t = 0$ and in the positive t-direction.

b.) Reparametrize $\mathbf{r}$ with respect to arc length.

c.) Find the curvature $\kappa(s)$ of $\mathbf{r}$.

a.) $\dot{\vec{r}}(t) = \langle \sin t, 1, -\cos t \rangle$

$$\|\dot{\vec{r}}(t)\| = \sqrt{\sin^2 t + 1 + \cos^2 t}$$
$$= \sqrt{1 + 1}$$
$$= \sqrt{2}$$

$$s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, u \Big|_0^t = \sqrt{2}\, t$$

$$s(t) = \sqrt{2}\, t$$

b.) Since $s = \sqrt{2}\, t$, then $t = \frac{s}{\sqrt{2}}$.

Then $\vec{r}(s) = \left\langle -\cos\left(\frac{s}{\sqrt{2}}\right), -\frac{s}{\sqrt{2}}, \sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$.

c.) $\vec{T}(s) = \frac{d\vec{r}}{ds} = \left\langle \frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right), \frac{-1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle$

$$\frac{d\vec{T}}{ds} = \left\langle \frac{1}{\sqrt{2}^2} \cos\left(\frac{s}{\sqrt{2}}\right), 0, \frac{1}{\sqrt{2}^2} \sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$\left\| \frac{d\vec{T}}{ds} \right\| = \sqrt{\left(\frac{1}{2}\right)^2 \cos^2\left(\frac{s}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{s}{\sqrt{2}}\right)}$$
$$= \sqrt{\left(\frac{1}{2}\right)^2 \left(\cos^2\left(\frac{s}{\sqrt{2}}\right) + \sin^2\left(\frac{s}{\sqrt{2}}\right)\right)}$$
$$= \sqrt{\frac{1}{4}} = \frac{1}{2}$$

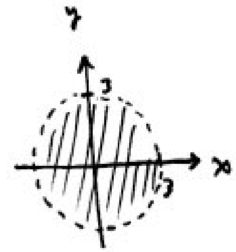
So, $\boxed{\kappa(s) = \frac{1}{2}}$

15. Consider the function $f(x, y) = \ln(9 - x^2 - y^2)$.

$a.$) State and sketch the domain of f.

$b.$) Find $\partial_x f$, $\partial_y f$, $\partial_{xy} f$, and $\partial_{yx} f$.

a.) Domain: $9 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 9$



b.) $\partial_x f = \dfrac{-2x}{9 - x^2 - y^2}$

$\partial_y f = \dfrac{-2y}{9 - x^2 - y^2}$

$\partial_{xx} f = \dfrac{(9 - x^2 - y^2)(-2) + 2x(-2x)}{(9 - x^2 - y^2)^2}$

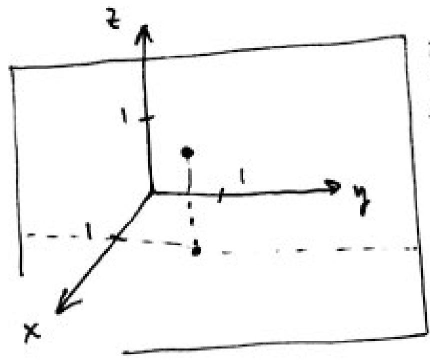
$= \dfrac{-18 - 2x^2 + 2y^2}{(9 - x^2 - y^2)^2}$

$\partial_{yy} f = \dfrac{-18 + 2x^2 - 2y^2}{(9 - x^2 - y^2)^2}$

$\partial_{xy} f = \partial_y(\partial_x f) = \dfrac{-4xy}{(9 - x^2 - y^2)^2}$

$\partial_{yx} f = \partial_x(\partial_y f) = \dfrac{-4xy}{(9 - x^2 - y^2)^2}$

16. Find a vector equation of the tangent line to the surface $f(x, y) = 2x^2 - y^2$ at the point $(1, 1, 1)$ that is parallel to the yz-plane. Show enough work.



Plane through P parallel to yz-plane.

The slope of the line is $\partial_y f(1,1)$.

$\partial_y f(x,y) = -2y$

$\partial_y f(1,1) = -2$.

$x = 1$ is fixed.

$y = t$

$z = 1 + (-2)(y - 1)$

$= 1 - 2y + 2$

$= -2y + 3$

so $z = -2t + 3$.

The line is given by:

$\boxed{\vec{r}(t) = \langle 1, t, -2t + 3 \rangle}$

17. Find the unit tangent, unit normal, and unit binormal vectors (**T**, **N**, and **B**) to the curve $\mathbf{r}(t) = \langle \cos t, t, -\sin t\rangle$ at the point $p(0, \frac{\pi}{2}, -1)$.

$t = \frac{\pi}{2}$

$$\dot{\vec{r}}(t) = \langle -\sin t, 1, -\cos t\rangle$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{\sin^2 t + 1 + \cos^2 t} = \sqrt{2}$$

$$\boxed{\vec{T}(t) = \left\langle -\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos t\right\rangle}$$

$$\dot{\vec{T}}(t) = \left\langle -\frac{1}{\sqrt{2}}\cos t, 0, \frac{1}{\sqrt{2}}\sin t\right\rangle$$

$\|\dot{\vec{T}}(t)\| = \frac{1}{\sqrt{2}}$, so

$$\boxed{\vec{N}(t) = \langle -\cos t, 0, \sin t\rangle}$$

$$\boxed{\begin{aligned}\vec{T}(p) &= \vec{T}(\pi/2) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle \\ \vec{N}(p) &= \vec{N}(\pi/2) = \langle 0, 0, 1\rangle\end{aligned}}$$

$$\vec{B}(p) = \vec{B}(\pi/2) = \vec{T}(\pi/2) \times \vec{N}(\pi/2)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\boxed{\vec{B}(p) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle}$$

18. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ at the point $p(3, 4)$.

$a.$) Find the linearization $L_p f(x, y)$ of f at p.

$b.$) Find the differential df_p at p.

$c.$) Use either the linearization or the differential to estimate the value of $\sqrt{3.01^2 + 3.99^2}$. Leave your answer as a reduced fraction.

a.) $\partial_x f = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow \partial_x f(p) = \frac{3}{5}$

$\partial_y f = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \partial_y f(p) = \frac{4}{5}$

$$\boxed{L_p f(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)}$$

b.) $df = \partial_x f dx + \partial_y f dy = \boxed{\frac{3}{5} dx + \frac{4}{5} dy = df}$

c.) $L_p f(3.01, 3.99) = 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.99-4)$

$= 5 + \frac{3}{5} \cdot \frac{1}{100} + \frac{4}{5} \cdot \frac{-1}{100}$

$= 5 - \frac{1}{500}$

$= 5 - \frac{2}{1000}$

$= \boxed{4.998}$

or $f(3.01, 3.99) \approx f(3,4) + df = 5 + \frac{3}{5} \cdot \frac{1}{100} + \frac{4}{5} \cdot \frac{-1}{100}$

and the rest is <u>exactly</u> the same.

$$\boxed{f(3.01, 3.99) \approx 4.998}$$

19. Find an equation of the normal plane to the space curve $\mathbf{r}(t) = \langle e^t, e^t\cos t, e^t\sin t\rangle$ at the point $p(1,1,0)$.

The normal vector to the normal plane is the tangent vector $\vec{T}$. (in the direction of $\vec{T}$...).

$\dot{\vec{r}}(t) = \langle e^t, e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t\rangle$

$\dot{\vec{r}}(0) = \langle 1, 1, 1\rangle$

$\dot{\vec{r}}(0)\cdot\vec{p} = 1\cdot 1 + 1\cdot 1 + 1\cdot 0 = 1+1+0 = 2.$

So, the normal plane is

$$\boxed{(x-1) + (y-1) + z + 2 = 0.}$$

20. Find the tangential and normal components of acceleration for the space curve $\mathbf{r}(t) = 3\cos t\,\mathbf{i} - 4\sin t\,\mathbf{j} + 359\,\mathbf{k}$.

$\vec{v}(t) = \dot{\vec{r}}(t) = \langle -3\sin t, -4\cos t, 0\rangle$

$\boxed{v(t) = \|\dot{\vec{r}}(t)\| = \sqrt{9\sin^2 t + 16\cos^2 t}}$

$\boxed{\dot{v}(t) = \dfrac{9\sin t + 16\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}}}$

$\ddot{\vec{r}}(t) = \langle -3\cos t, 4\sin t, 0\rangle$

$\dot{\vec{r}}(t)\times\ddot{\vec{r}}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin t & -4\cos t & 0 \\ -3\cos t & 4\sin t & 0 \end{vmatrix}$

$= \langle 0, 0, -12\rangle$

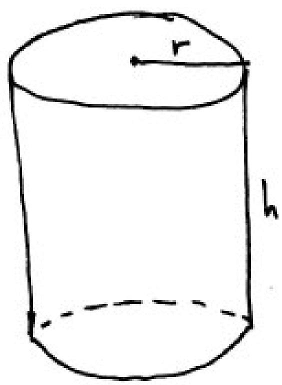
$\|\dot{\vec{r}}(t)\times\ddot{\vec{r}}(t)\| = 12.$

$\boxed{\kappa(t) = \dfrac{12}{\sqrt{9\sin^2 t + 16\cos^2 t}^{\,3}}}$

$\circledast\ \boxed{a_T(t) = \dot{v}(t) = \dfrac{9\sin t + 16\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}}}$

$a_N(t) = \kappa v^2 = \dfrac{12(9\sin^2 t + 16\cos^2 t)}{(9\sin^2 t + 16\cos^2 t)^{3/2}} = \boxed{\dfrac{12}{\sqrt{9\sin^2 t + 16\cos^2 t}} = a_N(t)}\ \circledast$

6. Done better:



$r = 4$ cm
$h = 12$ cm
$dr = dh = 0.04$ cm

$$\text{Volume} = V(r,h) = \pi r^2 h$$

$$\partial_r V = 2\pi r h$$

$$\partial_h V = \pi r^2$$

$$dV = \partial_r V \cdot dr + \partial_h V \cdot dh$$

$$dV = (2\pi r h)\,dr + (\pi r^2)\,dh$$

$$= (2\pi \cdot 4 \cdot 12)\left(\tfrac{1}{25}\right) + (\pi \cdot 16)\left(\tfrac{1}{25}\right)$$

$$= \frac{96\pi}{25} + \frac{16\pi}{25}$$

$$\boxed{dV = \frac{112\pi}{25}}$$

$$dV \approx 14.074 \text{ cm}^3$$

So, the amount of tin needed to build the can is approximately $\frac{112\pi}{25}$ cm^3,

or 14.074 cm^3.