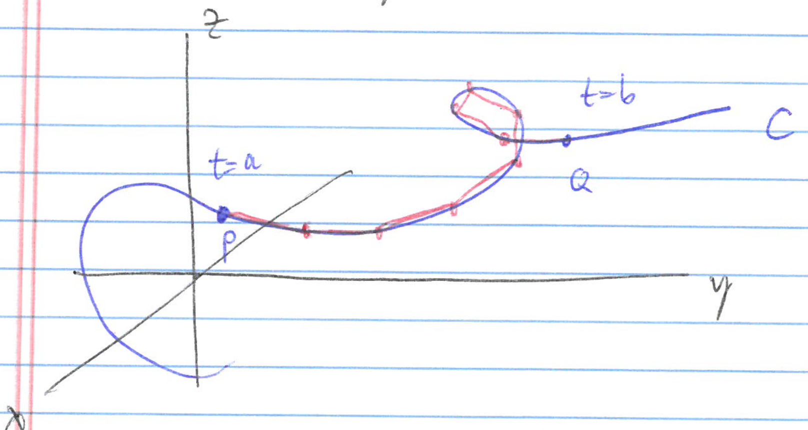
13.3 Arc Length and Curvature

Let $\vec{r}(t)$ be the vector function of a space curve C, so $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.



The arc legth of C between P and Q, where $P = \vec{r}(a)$ and $Q = \vec{r}(b)$ is given by

$$L = L(\vec{r}\,|\,[a,b]) = \int_a^b \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2}\, dt$$

In vector notation: $\boxed{L(\vec{r}\,|\,[a,b]) = \int_a^b \|\dot{\vec{r}}(t)\|\, dt}$

Ex. The helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from $P(1,0,0)$ to $Q(1,0,2\pi)$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\,\pi}.$$

Ex. Find the arc length function that measures the distance from $P(1,0,0)$ to any other pt $\vec{r}(t)$ on the helix.

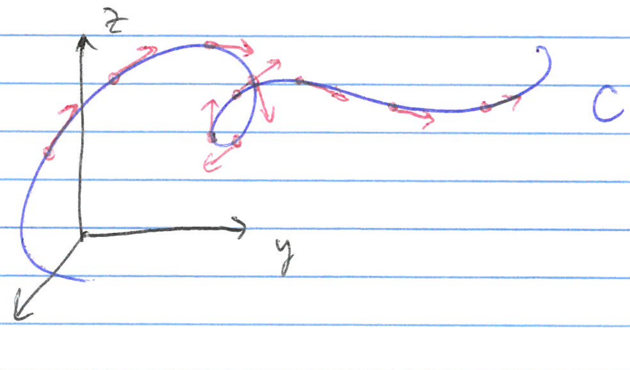
$$s(t) = L(\vec{r}\,|\,[a,t]) = \int_0^t \|\dot{\vec{r}}(u)\|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$$

So $s(t) = \sqrt{2}\, t \Rightarrow t = \frac{s}{\sqrt{2}}$

! You could then reparametrize the helix w/ respect to arc length : $\vec{r}(s) = \langle \cos(\frac{s}{\sqrt{2}}), \sin(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}} \rangle$

Notice that $\frac{ds}{dt} = \frac{d}{dt}\int_a^t \|\dot{\vec{r}}(u)\|\, du = \|\dot{\vec{r}}(t)\|$.

Curvature.



A parametrization $\vec{r}$ of a curve C is called smooth on an interval I if $\dot{\vec{r}}$ is continuous and $\dot{\vec{r}}(t) \neq \vec{0}$ on I.

A curve C is called smooth if it has a smooth parametrization.

A smooth curve has no sharp corners or cusps: when the tangent vectors move along C, they "turn" continuously.

Recall $\vec{T}(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|}$ the unit tangent vector field.

The tangent field represents the direction of the curve at any time t.

Defn. The curvature of a curve C is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

where $\vec{T}(s)$ is the unit tangent vector field, parametrized by arc length.

This measures the rate of change of the tangent vector ~~[illegible]~~ field with respect to arc length.

The curvature is much easier to compute if we can use and parameter t instead of having to use arc length.

Recall that $\quad \dfrac{ds}{dt} = \|\dot{\vec{r}}\| = \left\|\dfrac{d\vec{r}}{dt}\right\| \quad$ and $\quad \dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds} \cdot \dfrac{ds}{dt}$

Now $\quad \varkappa = \left\|\dfrac{d\vec{T}}{ds}\right\| = \left\|\dfrac{d\vec{T}/dt}{ds/dt}\right\| = \left\|\dfrac{\frac{d}{dt}(\vec{T})}{\|\dot{\vec{r}}(t)\|}\right\| = \dfrac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$

so $\quad \varkappa(s) = \left\|\dfrac{d\vec{T}}{ds}\right\| \quad$ and $\quad \varkappa(t) = \dfrac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$

<u>Ex.</u> Calculate the curvature of a circle of radius a.

$$\vec{r}(t) = \langle a\cos t, \ a\sin t\rangle$$

$$\dot{\vec{r}}(t) = \langle -a\sin t, \ a\cos t\rangle$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2(\sin^2 t + \cos^2 t)} = a$$

$$\vec{T}(t) = \langle -\sin t, \ \cos t\rangle$$

$$\dot{\vec{T}}(t) = \langle -\cos t, -\sin t\rangle$$

$$\|\dot{\vec{T}}(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

so $\varkappa(t) = \dfrac{1}{a}$ is constant.

Notice that as $a \to \infty$, $\varkappa \to 0$, and as $a \to 0$, $\varkappa \to \infty$.

Thus, circles of smaller radii are "more curved" than larger radius circles.

Thm. The curvature of the curve given by a vector function $\vec{r}(t)$ is

$$\kappa(t) = \frac{\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|^3}.$$

Proof. Recall that $\vec{T} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|}$ and $\|\dot{\vec{r}}\| = \frac{ds}{dt}$.

Then $\dot{\vec{r}} = \|\dot{\vec{r}}\|\, \vec{T} = \frac{ds}{dt} \vec{T}$,

and the product rule gives

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\dot{\vec{r}}\right) = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right) = \frac{d^2 s}{dt^2}\vec{T} + \frac{ds}{dt}\dot{\vec{T}}$$

Now

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \left(\frac{ds}{dt}\vec{T}\right) \times \left(\frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\dot{\vec{T}}\right)$$

$$= \underbrace{\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\left(\vec{T}\times\vec{T}\right)}_{=0} + \left(\frac{ds}{dt}\right)^2\left(\vec{T}\times\dot{\vec{T}}\right)$$

Now $\|\vec{T}\| = 1$ and $\vec{T} \perp \dot{\vec{T}}$ (prove it?!), so

$$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \left(\frac{ds}{dt}\right)^2 \|\vec{T}\times\dot{\vec{T}}\| = \left(\frac{ds}{dt}\right)^2 \|\vec{T}\|\,\|\dot{\vec{T}}\| \sin\left(\frac{\pi}{2}\right) = \left(\frac{ds}{dt}\right)^2 \|\dot{\vec{T}}\| = \|\dot{\vec{r}}\|^2\,\|\dot{\vec{T}}\|$$

So, $\|\dot{\vec{T}}\| = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^2}$

Then $\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3}$ $\square$

Ex. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ $\quad P(0,0,0)$

Find $\kappa(0)$ for this curve.

Get $\kappa(0) = 2$.

? Ex. (22) Find the curvature of $\vec{r}(t) = \langle t, t^2, e^t \rangle$

$$\dot{\vec{r}} = \langle 1, 2t, e^t \rangle$$
$$\ddot{\vec{r}} = \langle 0, 2, e^t \rangle$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & e^t \\ 0 & 2 & e^t \end{vmatrix} = \vec{i}(2te^t - 2e^t) - \vec{j}(e^t) + \vec{k}(2)$$

$$= \langle 2e^t(t-1), -e^t, 2 \rangle$$

$$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{4e^{2t}(t-1)^2 + e^{2t} + 4} = \sqrt{4t^2e^{2t} - 8te^{2t} + 4e^{2t} + e^{2t} + 4}$$

$$= \sqrt{4t^2e^{2t} - 8te^{2t} + 5e^{2t} + 4}$$

$$\|\dot{\vec{r}}\| = \sqrt{1 + 4t^2 + e^{2t}} \qquad \|\dot{\vec{r}}\|^3 = (1 + 4t^2 + e^{2t})^{3/2}$$

so $$\kappa(t) = \sqrt{\frac{4t^2e^{2t} - 8te^{2t} + 5e^{2t} + 4}{(1 + 4t^2 + e^{2t})^3}}.$$

Ex. Suppose C is a plane curve w/ equation $y = f(x)$. Choose x as the parameter and write

$$\vec{r}(x) = \langle x, f(x), 0 \rangle$$

Then $\dot{\vec{r}} = \langle 1, f'(x), 0 \rangle \qquad \|\dot{\vec{r}}\| = \sqrt{1 + (f'(x))^2}$

$\ddot{\vec{r}} = \langle 0, f''(x), 0 \rangle$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 0, 0, f''(x) \rangle \Rightarrow \|\dot{\vec{r}} \times \ddot{\vec{r}}\| = |f''(x)|$$

so $$\boxed{\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + (f'(x))^2}\right)^3}}$$

Ex. (28) Find the curvature of $y = x^4$

$$\kappa(x) = \frac{12x^2}{(1 + 16x^6)^{3/2}}$$

at $x = 1$

$$\kappa(1) = \frac{12}{17^{3/2}}$$

Ex.(28) $y = \tan x$ $\qquad f'(x) = \sec^2 x \qquad f''(x) = 2\sec x(\sec x \tan x)$

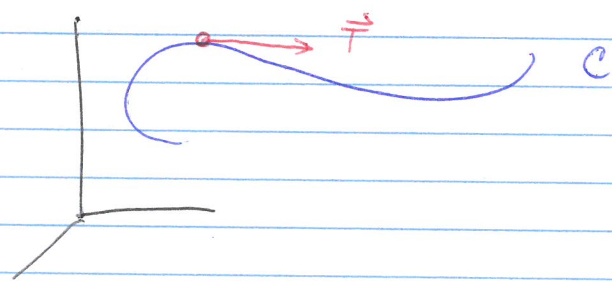
$$= 2\sec^2 x \tan x$$

so $\quad \kappa(x) = \dfrac{2\sec^2 x\,|\tan x|}{\left(\sqrt{1+\sec^2 x}\right)^3}$

$$\kappa(\pi/4) = \frac{2(\sqrt{2})^2\,|1|}{\left(\sqrt{1+\sqrt{2}^2}\right)^3} = \frac{4}{(\sqrt{5})^3}$$

The Normal and Binormal vectors

At any given point on a smooth space curve C, there are many vectors that are orthogonal to the unit tangent vector $\vec{T}$.



One can be "singled out" by observing that since $\|\vec{T}\| = 1$, then $\dot{\vec{T}} \perp \vec{T}$.

In general $\dot{\vec{T}}$ is not a unit vector, but at any point where $\kappa \neq 0$, then we can define the

Principal Unit Normal Vector $\vec{N}(t)$ as

$$\vec{N}(t) = \frac{\dot{\vec{T}}(t)}{\|\dot{\vec{T}}(t)\|}$$

We can then define the unit binormal vector $\vec{B}(t)$ to be

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

This gives us a moving frame of reference $\vec{T}, \vec{N}, \vec{B}$ at each point along $\vec{r}(t)$.

<u>Ex.</u> The helix again: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\dot{\vec{r}}(t) = \langle -\sin t, \cos t, 1 \rangle \qquad \|\dot{\vec{r}}\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$

so $\vec{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$

$\dot{\vec{T}} = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle \qquad \|\dot{\vec{T}}\| = \frac{1}{\sqrt{2}}$

$\dot{\vec{N}} = \langle -\cos t, -\sin t, 0 \rangle$

and $\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$

<u>Ex.</u> Find the $\vec{T}, \vec{N}, \vec{B}$ frame at the points $P(1,0,0)$ and $Q(0,1,\frac{\pi}{2})$ on the helix.

P: $\vec{T}(0) = \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \qquad Q$: $\vec{T}(\pi/2) = \langle \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

$\vec{N}(0) = \langle -1, 0, 0 \rangle \qquad \vec{N}(\pi/2) = \langle 0, -1, 0 \rangle$

$\vec{B}(0) = \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \qquad \vec{B}(\pi/2) = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

The plane determined by the normal and binormal vectors $\vec{N}$ and $\vec{B}$ at a point P is called the <u>normal plane</u> of C at P.

The plane determined by $\vec{T}$ and $\vec{N}$ is the <u>osculating plane</u> of C at P. This plane comes the closest to containing the part of the curve at P.

The circle that lies in the osculating plane of C at P, has the same tangent at P as C, lies on the concave side of C, and has radius $\rho = 1/\kappa$ is called the <u>osculating circle</u> of C at P.

This is the circle that best describes how C behaves at P: it has the same tangent, normal, and curvature.

Ex. Find the equations of the normal and osculating planes to the helix at $(0, 1, \pi/2)$.

- The normal plane has normal vector $\vec{T}(\pi/2) = \langle \frac{-1}{\sqrt{2}}, 0, \frac{+1}{\sqrt{2}} \rangle$

or just $\vec{r}\,'(\pi/2) = \langle -1, 0, 1 \rangle$.

so the eqn is $-1(x-0) + 0(y-1) + (z - \frac{\pi}{2}) = 0$

or $\boxed{-x + z = \frac{\pi}{2}}$

The osculating plane has normal vector $\vec{B}(\pi/2) = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$ or just $\vec{n} = \langle 1, 0, 1 \rangle$. So its eqn is

$$1(x-0) + 0(y-1) + 1(z - \pi/2) = 0$$

or $\boxed{x + z = \frac{\pi}{2}}$

Ex. Find the equation of the osculating circle to the parabola at the origin.

$$\kappa(x) = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}. \quad \text{so} \quad \kappa(0) = \frac{2}{1} = 2.$$

Thus the radius of the circle is $\frac{1}{2}$.

Its equation is $\boxed{x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}}$

or its vector equation is $\vec{r}(t) = \langle \frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t \rangle$