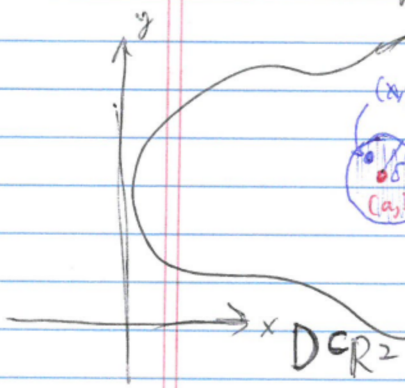
14.2 Limits and Continuity

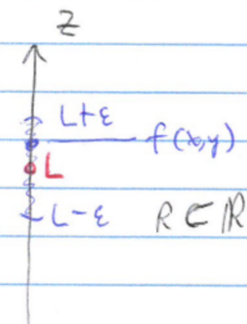
<u>Def'n</u>. Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the <u>limit of $f(x,y)$ as (x,y) approaches (a,b)</u> is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

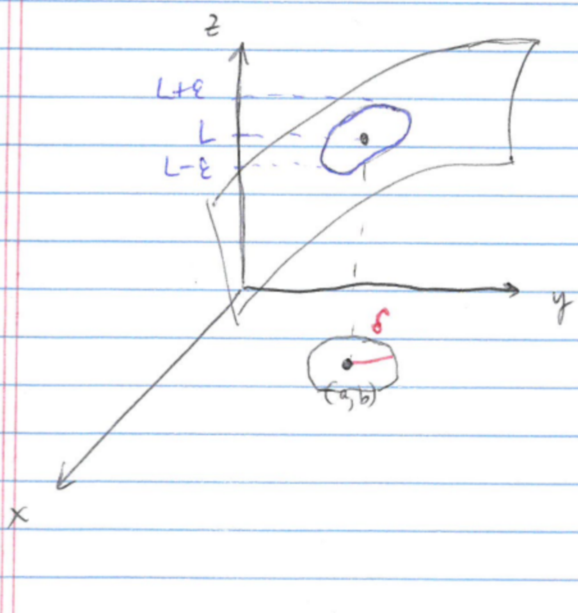
if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta, \quad \text{then } |f(x,y) - L| < \varepsilon.$$





$S = \Gamma(f)$



It's much easier to show that limits do <u>not</u> exist in this scenario than that they do.

$$\boxed{\text{If } f(x,y)\to L_1 \text{ as } (x,y)\to(a,b) \text{ along a path } C_1 \text{ and } f(x,y)\to L_2 \text{ as } (x,y)\to(a,b) \text{ along a second path } C_2, \text{ but } L_1 \neq L_2, \text{ then } \lim_{(x,y)\to(a,b)} f(x,y) = \text{DNE}}$$

Ex. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

- Take $C_1 = x$-axis and $C_2 = y$-axis.

Ex. $f(x,y) = \frac{\cancel{x}y}{x^2+y^2}$ Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

$\left.\begin{array}{l} C_1: \ x\text{- or } y\text{- axis} \\ C_2: \ y = x \end{array}\right\}$ do not agree.

Ex. $f(x,y) = \frac{xy^2}{x^2+y^4}$ $\lim_{(x,y)\to(0,0)} f(x,y) = DNE$

$C_1 : y = mx$ $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = 0$ for all m.

$C_2 : x = y^2$ $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \frac{1}{2} \neq 0$.

→ A limit that <u>does</u> exist:

Ex. $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Along x- or y- axes, $\lim = 0$. So we try to prove that the limit is 0 along <u>any</u> path by ε-δ methods.

Let $\varepsilon > 0$. We wish to find $\delta > 0$ such that if

$$0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2+y^2} \right| < \varepsilon$$

or if $0 < \sqrt{x^2+y^2} < \delta$ then $\frac{3x^2|y|}{x^2+y^2} < \varepsilon$

But $x^2 \le x^2 + y^2$ since $y^2 \ge 0$, so $\frac{x^2}{x^2+y^2} \le 1$

Therefore

$$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}.$$

Thus, if we choose $\delta = \varepsilon/3$, then

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \le 3\sqrt{x^2+y^2} < 3\delta = \cancel{3}\left(\frac{\varepsilon}{\cancel{3}}\right) = \varepsilon.$$

Hence, by the definition of limit,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0. \qquad \square$$

Continuity

Recall that a "Calc I" function of one variable is continuous at a point a iff

$$\lim_{x\to a} f(x) = f(a), \quad \text{or}$$

$$\boxed{\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right)} = f(a)$$

So a function is continuous iff it commutes w/ the limit.

Def'n. A function f of two variables is called continuous at the point $p(a,b)$ if and only if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$$\text{or} \quad \lim_{(x,y)\to(a,b)} f(x,y) = f\left(\lim_{(x,y)\to(a,b)} (x,y)\right) = f(p)$$

Crap!

We say f is <u>continuous</u> on D if f is continuous at every point of D.

<u>Theorem</u>. Polynomials and Rational Functions are continuous on their entire domains.

<u>Ex</u>. $\lim\limits_{(x,y)\to(1,2)} \left(x^2y^3 - x^3y^2 + 3x + 2y\right) = 11$

<u>Ex</u>. Where is $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

<u>Ex</u>. Let $g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Is g continuous?

<u>Ex</u>. Same for $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ and $g(x,y) = \begin{cases} f(x,y) & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Now g <u>is</u> continuous.

<u>Ex</u>. Where is $h(x,y) = \arctan(y/x)$ continuous?

$\longrightarrow$ Extend to functions of three or more variables:

Let f be defined on a subset of $\mathbb{R}^n \supset D$, and let f be defined in a neighborhood of $\vec{a} \in \mathbb{R}^n$.
Then

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x})$$

means that for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $\vec{x} \in D$ and $0 < \|\vec{x}-\vec{a}\| < \delta$, then

$$|f(\vec{x}) - L| < \varepsilon.$$