Name: Key
M344: Calculus III (Spring 2018)
Instructor: Justin Ryan
Unit I Exam: Chapters 12 and 13

WICHITA STATE UNIVERSITY

Read and follow all instructions.

Part I: True or False [2 point each]
Read each statement carefully. In the space provided, write **T** *if the statement is always true, or* **F** *otherwise.*

F **1.** The vector $\mathbf{x} = \langle 1,2,3\rangle$ is parallel to the plane $2x+4y+6z=-7$. perpendicular

T **2.** The arc length function of a space curve is defined by $s(t) = \int_0^t \|\dot{\mathbf{r}}(u)\|\,du$.

F **3.** The unit binormal vector is given by $\mathbf{B}(t) = \mathbf{N}(t)\times\mathbf{T}(t)$. $\vec{B} = \vec{T}\times\vec{N}$ (order matters)

F **4.** The parabola $y = ax^2$ has curvature $\kappa = 2a$ at the origin. $\kappa = 2|a|$

F **5.** The unit unit normal vector to a space curve at a point is given by $\mathbf{N}(s) = \kappa\dfrac{d\mathbf{T}}{ds}$. $\vec{N} = \dfrac{1}{\kappa}\dfrac{d\vec{T}}{ds}$

Part II: Multiple Choice [5 points each]
Write the letter corresponding to the appropriate answer in the space provided.

6 – 7. Consider the line parametrized by the vector function $\mathbf{r}(t) = \langle -t, 3-t, 2t+1\rangle$.

B **6.** What is the direction vector of the line?

A. $\langle 0,3,2\rangle$ **B.** $\langle -1,-1,2\rangle$
C. $\langle -1,3,2\rangle$ **D.** $\langle -1,-1,1\rangle$

C **7.** Find the distance between the line and the point $Q(0,0,1)$.

A. 3 **B.** $\dfrac{15}{2}$
C. $\dfrac{\sqrt{30}}{2}$ **D.** $\dfrac{\sqrt{15}}{2}$

8–11. Consider the vector function $\mathbf{r}(t) = \left\langle \frac{1}{1+t^2}, \sin(t), te^t \right\rangle$.

C ______ **8.** What is the domain of $\mathbf{r}$?

A. $(0, \infty)$ **B.** $(-\infty, 0)$

C. $(-\infty, \infty)$ **D.** $[0, \infty)$

A ______ **9.** Compute $\lim_{t\to 0} \mathbf{r}(t)$.

A. $\langle 1, 0, 0\rangle$ **B.** $\langle 1, 1, 1\rangle$

C. $\langle 0, 1, 1\rangle$ **D.** $\langle 1, 1, 0\rangle$

D ______ **10.** Compute $\dot{\mathbf{r}}(t)$.

A. $\left\langle \frac{-2t}{(1+t^2)^2}, -\cos t, e^t + te^t \right\rangle$ **B.** $\left\langle \frac{1}{2t}, \cos t, e^t \right\rangle$

C. $\left\langle \frac{-2t}{(1+t^2)^2}, \cos t, e^t \right\rangle$ **D.** $\left\langle \frac{-2t}{(1+t^2)^2}, \cos(t), (1+t)e^t \right\rangle$

A ______ **11.** Compute $\int \mathbf{r}(t)\, dt$.

A. $\left\langle \arctan(t), -\cos(t), (t-1)e^t \right\rangle + \mathbf{c}$

B. $\left\langle \arctan(t), \cos(t), (t-1)e^t \right\rangle + \mathbf{c}$

C. $\left\langle \frac{t}{t+\frac{1}{3}t^3}, -\cos(t), \frac{1}{2}t^2e^t \right\rangle + \mathbf{c}$

D. $\left\langle \frac{t}{t+\frac{1}{3}t^3}, -\cos(t), te^t - e^t \right\rangle + \mathbf{c}$

12–14. Consider the vector function $\mathbf{r}(t) = \langle -t, -\sin t, \cos t \rangle$.

C **12.** Find the arc length function $s(t)$ for $\mathbf{r}$ starting at $t = 0$ and moving in the positive t direction.

A. $s = \sqrt{2}$ **B.** $s = \sqrt{2t}$

C. $s = \sqrt{2}\,t$ **D.** $s = \dfrac{t}{\sqrt{2}}$

A **13.** Reparametrize $\mathbf{r}$ with respect to arc length.

A. $\mathbf{r}(s) = \left\langle -\dfrac{s}{\sqrt{2}}, -\sin\left(\dfrac{s}{\sqrt{2}}\right), \cos\left(\dfrac{s}{\sqrt{2}}\right) \right\rangle$

B. $\mathbf{r}(s) = \langle -s, -\sin(s), \cos(s) \rangle$

C. $\mathbf{r}(s) = \langle -1, -\cos(s), -\sin(s) \rangle$

D. $\mathbf{r}(s) = \left\langle -\dfrac{1}{\sqrt{2}}, -s\sin\left(\dfrac{t}{\sqrt{2}}\right), s\cos\left(\dfrac{t}{\sqrt{2}}\right) \right\rangle$

$\boxed{}$ **14.** Compute the curvature function $\kappa(s)$.

A. $\kappa(s) = \dfrac{s}{\sqrt{2}}$ **B.** $\kappa(s) = s\sqrt{2}$

C. $\kappa(s) = \sqrt{2}$ **D.** $\kappa(s) = \dfrac{1}{\sqrt{2}}$

$\vec{T}(s) = \left\langle \frac{-1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$

$\dfrac{d\vec{T}(s)}{ds} = \left\langle 0, \frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle$

$\left\| \dfrac{d\vec{T}}{ds} \right\| = \sqrt{\frac{1}{4}\sin^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{4}\cos^2\left(\frac{s}{\sqrt{2}}\right)}$

$= \sqrt{1/4} = 1/2$

Part III: Short Answer [10 points each]
You must complete all five problems in this part. Show enough work. Clearly mark yo ur final answers. Partial credit given when deserved.

15. Consider the space curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at $t = \pi$.

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\dot{\vec{r}}(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\vec{T}(t) = \left\langle -\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \right\rangle$ $\boxed{\vec{T}(\pi) = \left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$

$\dot{\vec{T}}(t) = \left\langle -\frac{1}{\sqrt{2}} \cos t, \frac{-1}{\sqrt{2}} \sin t, 0 \right\rangle$

$\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$ $\boxed{\vec{N}(\pi) = \langle 1, 0, 0 \rangle}$

$\boxed{\vec{B}(\pi) = \vec{T}(\pi) \times \vec{N}(\pi) = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$

16. Show that the curvature of a circle of radius a is constant, $\kappa = \frac{1}{a}$.

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t\rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$s(t) = \int_0^t a\,du = at \Rightarrow t = \frac{s}{a}$

$\vec{r}(s) = \langle a\cos(\frac{s}{a}), a\sin(\frac{s}{a})\rangle$

$\vec{T} = \frac{d\vec{r}}{ds} = \langle -\sin(\frac{s}{a}), \cos(\frac{s}{a})\rangle$

$\frac{d\vec{T}}{ds} = \langle -\frac{1}{a}\cos(\frac{s}{a}), \frac{-1}{a}\sin(\frac{s}{a})\rangle$

$\kappa(s) = \left\|\frac{d\vec{T}}{ds}\right\| = \sqrt{\frac{1}{a^2}\cos^2(\frac{s}{a}) + \frac{1}{a^2}\sin^2(\frac{s}{a})}$

$= \sqrt{1/a^2} = \frac{1}{a}.$

17. Find an equation of the osculating circle to the curve $y = 4x^2 - x^4$ at the origin, $(0,0)$.

$\kappa(0) = \frac{|f''(0)|}{\sqrt{1+f'(0)^2}^{\,3}} = \frac{8}{1} = 8$ So, $r = \frac{1}{\kappa} = \frac{1}{8}$.

$f'(x) = 8x - 4x^3$ $\quad f'(0) = 0.$

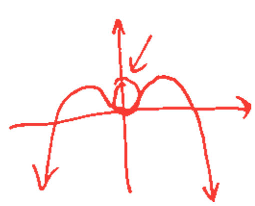
$f''(x) = 8 - 12x^2$ $\quad f''(0) = 8$

The circle is thus

$$\boxed{x^2 + (y - 1/8)^2 = \frac{1}{64}}$$

$f''(0) > 0$ implies the curve is concave up and $f'(0) = 0$ means $x=0$ is a local minimum.

Hence $\vec{N} = \langle 0, 1\rangle$.



18. Prove that the unit tangent vector field $\mathbf{T}$ of a smooth space curve satisfies $\dot{\mathbf{T}}(t) \perp \mathbf{T}(t)$ for all t in the domain of $\mathbf{T}$.

Since $\vec{T}$ is a unit vector field, $\|\vec{T}\|^2 = 1$.
This implies that $\vec{T} \cdot \vec{T} = 1$, hence

$$\frac{d}{dt}(\vec{T} \cdot \vec{T} = 1) \Rightarrow \dot{\vec{T}} \cdot \vec{T} + \vec{T} \cdot \dot{\vec{T}} = 0$$
$$\Rightarrow 2\dot{\vec{T}} \cdot \vec{T} = 0$$
$$\Rightarrow \dot{\vec{T}} \cdot \vec{T} = 0$$
$$\Rightarrow \dot{\vec{T}} \perp \vec{T}. \quad \square$$

19. Find an equation of the plane passing through the points $P(1,0,0)$, $Q(0,2,0)$, and $R(0,0,3)$.

$\vec{PQ} = \langle -1, 2, 0 \rangle$

$\vec{PR} = \langle -1, 0, 3 \rangle$

$\vec{n} = \langle 6, 3, 2 \rangle$

$\vec{n} \cdot \vec{P} = 6$

$$\boxed{6x + 3y + 2z = 6}$$