M555: Differential Equations Chapter 4 Concept check

1. characteristic polynomial: $r^4 - 1 = 0$

$$(r^2+1)(r^2-1) = 0$$

$$r = \pm i \quad r = \pm 1$$

So, $\boxed{y(t) = c_1 \cos t + c_2 \sin t + c_3 e^{t} + c_4 e^{-t}}$

2. Recall: $\begin{cases} \cosh(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ \sinh(t) = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \end{cases} \Rightarrow \begin{cases} \cosh(t) + \sinh(t) = e^{t} \\ \cosh(t) - \sinh(t) = e^{-t} \end{cases}$

Then,

$$\begin{aligned} y(t) &= c_1 \cos t + c_2 \sin t + c_3(\cosh t + \sinh t) + c_4(\cosh t - \sinh t) \\ &= c_1 \cos t + c_2 \sin t + (c_3 + c_4)\cosh t + ((c_3 - c_4)\sinh t \\ &= c_1 \cos t + c_2 \sin t + c_5 \cosh(t) + c_6 \sinh(t) \quad \checkmark \end{aligned}$$

3.
$$\left.\begin{aligned} y &= c_1 \cos t + c_2 \sin t + c_5 \cosh t + c_6 \sinh t \\ y' &= -c_1 \sin t + c_2 \cos t + c_5 \sinh t + c_6 \cosh t \\ y'' &= -c_1 \cos t - c_2 \sin t + c_5 \cosh t + c_6 \sinh t \\ y''' &= c_1 \sin t - c_2 \cos t + c_6 \sinh t + c_6 \cosh t \\ y^{(4)} &= c_1 \cos t + c_2 \sin t + c_5 \cosh t + c_6 \sinh t \end{aligned}\right\} \quad \begin{aligned} y(0) &= c_1 + c_5 = 0 \\ y'(0) &= c_2 + c_6 = 0 \\ y''(0) &= -c_1 + c_5 = 1 \\ y'''(0) &= -c_2 + c_6 = 1 \end{aligned} \quad (*)$$

$2c_5 = 1 \Rightarrow c_5 = 1/2,\ c_1 = -1/2, \quad 2c_6 = 1 \Rightarrow c_6 = 1/2,\ c_2 = -1/2.$

$$\boxed{y(t) = -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\cosh t + \frac{1}{2}\sinh t}$$

Using sinht and cosht make the system of equations (*) turn out much nicer than using e^{t} and e^{-t}.

4. $y''' - y' = t$

$r^3 - r = 0$

$r(r^2-1) = 0$

$r(r-1)(r+1) = 0$

$y_h(t) = C_1 + C_2 e^t + C_3 e^{-t}$

Undetermined coefficients:

Guess $Y(t) = At + B$

but B is a homog. sol'n. So, guess instead

$Y(t) = At^2 + Bt$

$Y' = 2At + B$

$Y'' = 2A$

$Y''' = 0$

$Y''' - Y' = -(2At+B) = t$

so $-2A = 1 \rightarrow A = -1/2$

and $B = 0$.

The general solution is $\boxed{y(t) = C_1 + C_2 e^t + C_3 e^{-t} - \frac{1}{2}t^2}$

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5. $y''' - y'' + y' - y = \sec t$

$r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + 1(r-1) = 0$

$(r^2+1)(r-1) = 0$

$r = \pm i,\ r = 1$

$y_1 = \cos t$

$y_2 = \sin t$

$y_3 = e^t$

$$W(y_1, y_2, y_3) = \begin{vmatrix} \cos t & \sin t & e^t \\ -\sin t & \cos t & e^t \\ -\cos t & -\sin t & e^t \end{vmatrix}$$

$$= e^t(\sin^2 t + \cos^2 t) - e^t(-\cos t \sin t + \cos t \sin t) + e^t(\cos^2 t + \sin^2 t)$$

$$= 2e^t$$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

$$W(y_1, y_3) = \begin{vmatrix} \cos t & e^t \\ -\sin t & e^t \end{vmatrix} = e^t(\cos t + \sin t)$$

$$W(y_2, y_3) = \begin{vmatrix} \sin t & e^t \\ \cos t & e^t \end{vmatrix} = e^t(\sin t - \cos t)$$

Variation of Parameters says the solution is:

$$y(t) = y_1(t) \underbrace{\int \frac{W(y_2,y_3)\, g(t)}{W(y_1,y_2,y_3)}\, dt}_{I_1} - y_2(t) \underbrace{\int \frac{W(y_1,y_3)\, g(t)}{W(y_1,y_2,y_3)}\, dt}_{I_2} + y_3(t) \underbrace{\int \frac{W(y_1,y_2)\, g(t)}{W(y_1,y_2,y_3)}\, dt}_{I_3}$$

$$I_1 = \int \frac{e^t(\sin t - \cos t)\sec t}{2e^t}\, dt = \frac{1}{2}\int \frac{\sin t}{\cos t} - 1\, dt = \frac{1}{2}\ln|\sec t| - \frac{1}{2}t + c_1$$

$$I_2 = \int \frac{e^t(\sin t + \cos t)\sec t}{2e^t}\, dt = \frac{1}{2}\int \frac{\sin t}{\cos t} + 1\, dt = \frac{1}{2}\ln|\sec t| + \frac{1}{2}t + c_2$$

$$I_3 = \int \frac{1 \sec t}{2e^t}\, dt = \frac{1}{2}\int e^{-t}\sec t\, dt$$ This is not nice! So leave it.

Then the solution is

$$y(t) = \cos t\left(\frac{1}{2}\ln|\sec t| - \frac{1}{2}t + C_1\right) - \sin t\left(\frac{1}{2}\ln|\sec t| + \frac{1}{2}t + C_2\right) + e^t\left(\frac{1}{2}\int e^{-t}\sec t\, dt + C_3\right)$$

or

$$y(t) = C_1\cos t + C_2\sin t + C_3 e^t + \frac{1}{2}\cos t\ln|\sec t| - \frac{1}{2}\sin t\ln|\sec t| - \frac{1}{2}t\cos t - \frac{1}{2}t\sin t + \frac{e^t}{2}\int e^{-t}\sec t\, dt$$