M555 - Concept Check: Ch 3 & 4

1. $y''' - 2y'' - y' + 2y = e^{4t}$

char. eq'n: $r^3 - 2r^2 - r + 2 = 0$

$r^2(r-2) - 1(r-2) = 0$

$(r+1)(r-1)(r-2) = 0$

$\left.\begin{array}{l} y_1 = e^{-t} \\ y_2 = e^{t} \\ y_3 = e^{2t} \end{array}\right\}$ homogeneous sol'ns

Undetermined Coeff:

$Y(t) = Ae^{4t}$

$Y'(t) = 4Ae^{4t}$

$Y''(t) = 16Ae^{4t}$

$Y'''(t) = 64Ae^{4t}$

$Y''' - 2Y'' - Y' + 2Y = (64A - 32A - 4A + 2A)e^{4t} = e^{4t}$

$\Rightarrow 30A = 1 \Rightarrow A = 1/30.$

so the general solution is $\boxed{\varphi(t) = C_1 e^{-t} + C_2 e^{t} + C_3 e^{2t} + \frac{1}{30} e^{4t}}$

2. Reduction of Order:

$\varphi(x) = v(x) y_1(x) = v(x) \sin(x^2)$

$\varphi'(x) = v'(x)\sin(x^2) + 2x\, v(x)\cos(x^2)$

$\varphi''(x) = v''(x)\sin(x^2) + 4x\, v'(x)\cos(x^2) - 4x^2 v(x)\sin(x^2) + 2v(x)\cos(x^2)$

Plug in to the DE:

$x\varphi'' - \varphi' + 4x^3\varphi = x\left(v''\sin(x^2) + 4x\,v'\cos(x^2) - 4x^2 v\sin(x^2) + 2v\cos(x^2)\right) - \left(v'\sin(x^2) + 2x\,v\cos(x^2)\right)$

$+ 4x^3\left(v\sin(x^2)\right)$

$= v''\left(x\sin(x^2)\right) + v'\left(4x^2\cos(x^2) - \sin(x^2)\right) + v\left(\cancel{-4x^3\sin(x^2)} + \cancel{2xv\cos(x^2)} - \cancel{2xv\cos(x^2)} + \cancel{4x^3\sin(x^2)}\right)$

$\Rightarrow v''\left(x\sin(x^2)\right) + v'\left(4x^2\cos(x^2) - \sin(x^2)\right) = 0$

This is a first order separable equation for v'.

$$\frac{d(v')}{dx} = \frac{\sin(x^2) - 4x^2\cos(x^2)}{x\sin(x^2)}\, v'$$

$$\Rightarrow \frac{1}{v'}\, d(v') = \left(\frac{1}{x} - 4x\cot(x^2)\right) dx$$

$$\ln(v') = \ln x + 2\ln(\csc(x^2)) + C$$

$\int x\cot(x^2)\,dx$

$u = x^2 \quad du = 2x\,dx$

$= \frac{1}{2}\int \cot(u)\,du$

$= -\frac{1}{2}\ln|\csc u|$

2. cont'd

$$\ln(v') = \ln x + 2\ln\left(\csc(x^2)\right) + C$$
$$= \ln x + \ln\left(\csc^2(x^2)\right) + C$$

$\Rightarrow \quad v'(x) = C_1 x \csc^2(x^2) \quad$ or $\quad \frac{dv}{dx} = C_1 x \csc^2(x^2)$

$$v = \int C_1 x \csc^2(x^2)\,dx = \int C_1 \csc^2(u)\,du = -C_1 \cot u + C_2$$
$u = x^2 \quad du = 2x\,dx$

so, $v(x) = C_1 \cot(x^2) + C_2$ and the general sol'n of the DE is

$$\varphi(x) = \left(C_1 \cot(x^2) + C_2\right)\sin(x^2) = C_1 \cot(x^2)\sin(x^2) + C_2 \sin(x^2)$$

or

$$\boxed{\varphi(x) = C_1 \cos(x^2) + C_2 \sin(x^2)}$$

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3. $y'' - 2y' + y = \frac{e^t}{1+t^2}$

Char. eq'n: $r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$
$r = 1, 1$

$y_1 = e^t$
$y_2 = te^t$

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$
$$= e^{2t} + te^{2t} - e^{2t}$$
$$= te^{2t}$$

ERROR :(

See pg. 3.

Variation of Parameters:

$$y(t) = -y_1(t)\underbrace{\int \frac{y_2(t)\,g(t)}{W(t)}\,dt}_{I_1} + y_2(t)\underbrace{\int \frac{y_1(t)\,g(t)}{W(t)}\,dt}_{I_2}$$

$$I_1 = \int \frac{te^t \cdot e^t}{te^{2t}(1+t^2)}\,dt = \int \frac{1}{1+t^2}\,dt = \arctan(t) + C_1$$

$$I_2 = \int \frac{e^t\, e^t}{te^{2t}(1+t^2)}\,dt = \int \frac{1}{t(1+t^2)}\,dt = \int \frac{1}{t} - \frac{1}{2}\frac{2t}{1+t^2}\,dt = \ln t - \frac{1}{2}\ln(1+t^2) + C_2$$

PFD: $\frac{1}{t(1+t^2)} = \frac{A}{t} + \frac{Bt+C}{1+t^2}$

$\Rightarrow 1 = A(1+t^2) + (Bt+C)t = A + Ct + (A+B)t^2$

$$\begin{cases} A = 1 \\ C = 0 \\ B = -A = -1 \end{cases}$$

so the general solution is:

$$\boxed{y(t) = -e^t \arctan(t) + te^t \ln t - \frac{1}{2}t\,e^t \ln(1+t^2) + C_1 e^t + C_2 te^t}$$

Concept Check: Ch 3/4 ADDENDUM/ERRATUM

3. $W(y_1, y_2)$ should be e^{2t}, not te^{2t}.

I_1 and I_2 then need to be re-computed.

$$I_1 = \int \frac{te^t \cdot e^t}{e^{2t}(1+t^2)}\,dt = \int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{2t}{1+t^2}\,dt = \frac{1}{2}\ln|1+t^2| + C_1$$

$$I_2 = \int \frac{e^t \cdot e^t}{e^{2t}(1+t^2)}\,dt = \int \frac{1}{1+t^2}\,dt = \arctan(t) + C_2$$

and the general solution is:

$$\boxed{y(t) = -\frac{e^t}{2}\ln(1+t^2) + te^t\arctan(t) + C_1e^t + C_2te^t}$$