M555: Differential Equations   Unit I Review Solutions

1. $\frac{dy}{dt} = 2y - 5 = 2\left(y - \frac{5}{2}\right)$

$\int \frac{1}{y - 5/2}\,dy = \int 2\,dt$

$\ln(y - 5/2) = 2t + C$

$y - 5/2 = Ce^{2t}$

$y = Ce^{2t} + 5/2$

$y(0) = C + 5/2 = y_0 \Rightarrow C = y_0 - 5/2$

So,

$\boxed{y = (y_0 - 5/2)e^{2t} + 5/2}$

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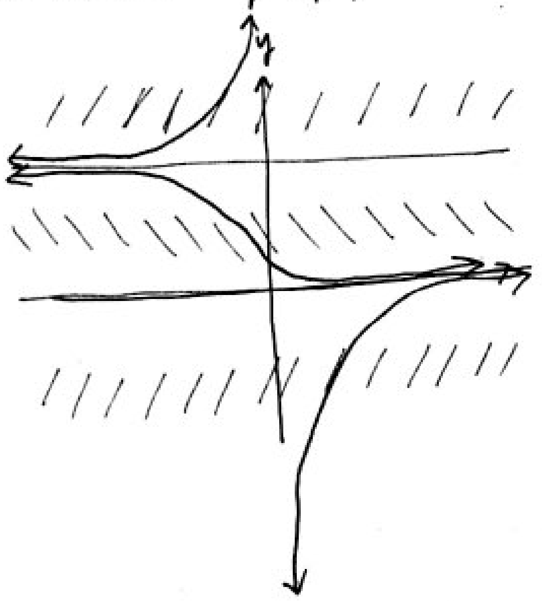
2. $y' = -y(5-y) = y(y-5)$

Equilibrium solutions: $y=0$, $y=5$



$y=6$, $y'=6$

$y=5$, $y'=0$

$y=2$, $y'=-6$

$y=0$, $y'=0$

$y=-1$, $y'=6$

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3. $\begin{cases} y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \\ y(1) = 1/2 \end{cases}$

$\int \frac{2}{t}\,dt = 2\ln t \Rightarrow \mu(t) = t^2.$

$y = \frac{1}{t^2}\left(\int_1^t u^2\left(u - 1 + \frac{1}{u}\right)du + 1/2\right)$

$= \frac{1}{t^2}\left(\int_1^t u^3 - u^2 + u\,du + 1/2\right)$

$= \frac{1}{t^2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 - \underbrace{(1/4 - 1/3 + 1/2)}_{-1/12} + 1/2\right)$

$\boxed{y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}}, \; t > 0$

$p(t) = \frac{2}{t}$ is continuous for $t \neq 0$

$g(t) = t - 1 + \frac{1}{t}$ is continuous for $t \neq 0$.

The FEUT says the DE has a sol'n w/ domain $(0, \infty)$.

4. $\begin{cases} y' + \frac{2}{t} y = \frac{\sin t}{t} \\ t > 0 \end{cases}$

$e^{-\int \frac{2}{t} dt} = e^{-\ln t^2} = t^{-2}$

$y = A t^{-2}$

$y' = A' t^{-2} + A(-2t^{-3})$

$A' t^{-2} + \underbrace{A(-2t^{-3}) + \frac{2}{t} A t^{-2}}_{=0} = \frac{\sin t}{t}$

$A' t^{-2} = \frac{\sin t}{t}$

$\Rightarrow A' = \frac{\sin t}{t} t^2$

$A = \int \left(\frac{\sin t}{t}\right) t \, dt = -t\cos t + \sin t + C$

So, $y(t) = t^{-2}(-t\cos t + \sin t + C)$

or

$$\boxed{y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}}$$

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5. $f(x,y) = \frac{1-2x}{y}$ is continuous for $y \neq 0$.

$\frac{\partial f}{\partial y} = \frac{2x-1}{y^2}$ is continuous for $y \neq 0$. ✓

$\frac{dy}{dx} = \frac{1-2x}{y} \Rightarrow \int y \, dy = \int (1-2x) \, dx$

$\Rightarrow \frac{1}{2} y^2 = x - x^2 + C$

$\Rightarrow y^2 = 2x - 2x^2 + C$

$\Rightarrow y = \pm\sqrt{2x - 2x^2 + C}$

$y(1) = \pm\sqrt{2 - 2 + C} = -2 \Rightarrow C = 4$

The solution is thus,

$$\boxed{y(x) = -\sqrt{2x - 2x^2 + 4}}$$

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6. $y = xv$ and the DE becomes

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\cancel{v} + x\frac{dv}{dx} = \frac{x^2 + x^2 v + x^2 v^2}{x^2} = v^2 + \cancel{v}$

or $x\frac{dv}{dx} = v^2 \Rightarrow \frac{1}{v^2} dv = \frac{1}{x} dx \Rightarrow \frac{-1}{v} = \ln x + C$

therefore $v(x) = \frac{-1}{\ln x + C}$ and $\boxed{y = \frac{-x}{\ln x + C}}$

7. $f(x,y) = \dfrac{9x^2+y-1}{4y-x}$ continuous if $x \neq 4y$

$\dfrac{\partial f}{\partial y} = \dfrac{(4y-x) - (9x^2+y-1)(4)}{(4y-x)^2}$ also continuous if $x \neq 4y$

Initial condition is $(x,y) = (1,0)$ ✓

$$\left.\begin{aligned} M(x,y) &= 9x^2+y-1 & \frac{\partial M}{\partial y} &= 1 \\ N(x,y) &= -(4y-x) & \frac{\partial N}{\partial x} &= -(-1) = 1 \end{aligned}\right\} \text{Exact!}$$

$$\left.\begin{aligned} \varphi &= \int 9x^2+y-1 \, dx = 3x^3+xy-x + C_1(y) \\ \varphi &= \int x-4y \, dy = xy-2y^2 + C_2(x) \end{aligned}\right\}$$

The solution is $3x^3+xy-x-2y^2 = C$

Plugging in the initial data: $3(1)^3 + 1(0) - 1 - 2(0)^2 = C$

$\Rightarrow C = 2.$

So the solution is given by $-2y^2 + xy + 3x^3 - x - 2 = 0$

$$-2\left(y^2 - \frac{x}{2}y \quad\right) = -3x^3+x+2$$

$$\left(y-\frac{x}{4}\right)^2 = 2\left(3x^3-x-2\right) + \frac{x^2}{16}$$

then $y = \dfrac{x}{4} \pm \sqrt{6x^3 + \frac{1}{16}x^2 - 2x - 4}$

For the initial condition to be satisfied, the solution must be

$$\boxed{y = \frac{x}{4} + \sqrt{6x^3+\frac{1}{16}x^2-2x-4}}$$

8. $M(x,y) = (x+2)\sin y \qquad \partial M/\partial y = +(x+2)\cos y$
$N(x,y) = x\cos y \qquad \partial N/\partial x = \cos y$
} Not exact.

$\mu M = (x^2e^x + 2xe^x)\sin y \qquad \partial(\mu M)/\partial y = +(x^2e^x+2xe^x)\cos y$
$\mu N = x^2e^x\cos y \qquad \partial(\mu N)/\partial x = (x^2e^x + 2xe^x)\cos y$
} Exact. ☺

$\varphi = \int \mu M\,dx = \int (x^2e^x+2xe^x)\sin y\,dx = \sin y \int d[x^2e^x] = x^2e^x\sin y + C_1(y)$

$\varphi = \int \mu N\,dy = \int x^2e^x\cos y\,dy = x^2e^x\sin y + C_2(x)$

So the general solution is $x^2e^x\sin y = C$

or $\sin y = \frac{C}{x^2e^x}$ or $\boxed{y = \arcsin\left(\frac{C}{x^2e^x}\right)}$

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9. $y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$

$p(t) = \frac{2t}{4-t^2}$, $g(t) = \frac{3t^2}{4-t^2}$, continuous for $t \neq \pm 2$.

since the initial t-value is $t=1$, the domain of the solution is $(-2,2)$.

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10. Next page.

10. $y' = -\frac{1}{2}y + t$

$f(t,y) = -\frac{1}{2}y + t$

$\varphi_0 = 0$

$\varphi_1 = \int_0^t -\frac{1}{2}0 + u\,du = \frac{1}{2}u^2\Big|_0^t = \frac{1}{2}t^2$

$\varphi_2 = \int_0^t -\frac{1}{4}u^2 + u\,du = -\frac{1}{12}u^3 + \frac{1}{2}u^2\Big|_0^t = -\frac{1}{12}t^3 + \frac{1}{2}t^2$

$\varphi_3 = \int_0^t \frac{1}{24}u^3 - \frac{1}{4}u^2 + u\,du = \frac{1}{2^{2}4!}t^4 - \frac{1}{2^{1}3!}t^3 + \frac{1}{2}t^2$

$\varphi_n = \sum_{i=1}^{n} \frac{1}{2^{i-1}(i+1)!}t^{i+1}$ and $\boxed{\varphi(t) = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}(i+1)!}t^{i+1}}$