M555 - Unit II Review Guide

1. Calculate the wronskians for each set. If it is 0, then the functions are linearly dependent; otherwise they are independent.

a.) $W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \neq 0.$ Linearly independent.

b.) $W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = 0$ Linearly dependent

c.) $W(\sin t, \cos t) = \begin{vmatrix} \sin t & \cos t \\ +\cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -(\sin^2 t + \cos^2 t) = -1 \neq 0.$ Linearly independent

d.) $W(t, te^t, t^2e^t) = \begin{vmatrix} t & te^t & t^2e^t \\ 1 & (1+t)e^t & (t^2+2t)e^t \\ 0 & (2+t)e^t & (t^2+4t+2)e^t \end{vmatrix}$

$= t\begin{vmatrix} (1+t)e^t & (t^2+2t)e^t \\ (2+t)e^t & (t^2+4t+2)e^t \end{vmatrix} - \begin{vmatrix} te^t & t^2e^t \\ (2+t)e^t & (t^2+4t+2)e^t \end{vmatrix}$

$= e^{2t}\left[t(1+t)(t^2+4t+2) - t(2+t)(t^2+2t) - t(t^2+4t+2) + t^2(2+t)\right]$

$= e^{2t}\left[(\cancel{t^3} + \cancel{4t^2} + \cancel{2t} + \cancel{t^4} + \cancel{4t^3} + \cancel{2t^2}) - (\cancel{4t^3} + \cancel{4t^2} + \cancel{t^4}) - (\cancel{t^3} + \cancel{4t^2} + \cancel{2t}) + (\cancel{2t^2} + t^3)\right]$

$= t^3e^{2t} \neq 0$ Linearly independent.

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2. Solve the characteristic equations and write the solutions.

a.) $r^2 + 8r - 9 = 0$

$(r+9)(r-1) = 0$

$r = -9, r = 1$

$\boxed{y(t) = C_1e^{-9t} + C_2e^t}$

b.) $r^2 - 2r + 6 = 0$

$r^2 - 2r + 1 = -6 + 1$

$(r-1)^2 = -5$

$r = 1 \pm \sqrt{5}\, i$

$\boxed{y(t) = e^t\left(C_1\cos(\sqrt{5}\,t) + C_2\sin(\sqrt{5}\,t)\right)}$

c.) $25r^2 - 20r + 4 = 0$

$(5r-2)^2 = 0$

$r = \frac{2}{5}, \frac{2}{5}$

$\boxed{y(t) = C_1e^{\frac{2}{5}t} + C_2te^{\frac{2}{5}t}}$

3. Reduction of Order:

$\varphi(x) = v(x)e^x$

$\varphi' = v'e^x + ve^x$

$\varphi'' = v''e^x + 2v'e^x + ve^x$

Plug in to DE:

$$(x-1)\varphi'' - x\varphi' + \varphi = (x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x$$

$$= e^x v''(x-1) + e^x v'(2(x-1) - x) + e^x v(\overset{0}{\cancel{(x-1) - x + 1}}) = 0$$

$$= e^x\left((x-1)v'' + (x-2)v'\right) = 0$$

We obtain the first order separable DE for v':

$$v'' = \frac{d(v')}{dx} = \frac{-(x-2)}{x-1} v'$$

$$\Rightarrow \frac{1}{v'} d(v') = \frac{-(x-1)}{x-1} + \frac{1}{x-1}\, dx$$

$$\Rightarrow \ln(v') = -x + \ln(x-1) + C_1$$

$$\Rightarrow v' = C_1(x-1)e^{-x}$$

$$\Rightarrow v(x) = \int C_1(x-1)e^{-x}\,dx = C_1\left(-(x-1)e^{-x} - e^{-x}\right) + C_2$$

$$= C_1(x-1)e^{-x} + C_1e^{-x} + C_2$$

The general solution is thus

$$\varphi(x) = e^x\left(C_1(x-1)e^{-x} + C_1e^{-x} + C_2\right) = C_1(x-1) + C_1 + C_2e^x = C_1x + C_2e^x$$

$$\boxed{\varphi(x) = C_1x + C_2e^x}$$

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4. Euler Equation

$\varphi = t^r$

$\varphi' = rt^{r-1}$

$\varphi'' = r(r-1)t^{r-2}$

Plug in:

$t^2\varphi'' - 4t\varphi' + 4\varphi$

$= t^2 r(r-1)t^{r-2} - 4tr\,t^{r-1} + 4t^r$

$= t^r(r^2 - r - 4r + 4) = 0$

$r^2 - 5r + 4 = (r-4)(r-1) = 0$

$r = 1, 4$

So the solution is:

$$\boxed{\varphi(t) = C_1t + C_2t^4}$$

5. $\begin{cases} y'' - 2y' + y = te^t + 4 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

<u>Homog:</u> $r^2 - 2r + 1 = (r-1)^2 = 0$

$r = 1, 1$

$y_1 = e^t$

$y_2 = te^t$

Guess:

$$Y(t) = t^2(At+B)e^t + C = At^3e^t + Bt^2e^t + C = (At^3 + Bt^2)e^t + C$$

$$Y' = 3At^2e^t + At^3e^t + 2Bte^t + Bt^2e^t = \left(At^3 + (3A+B)t^2 + 2Bt\right)e^t$$

$$Y'' = \left(3At^2 + 2(3A+B)t + 2B\right)e^t + \left(At^3 + (3A+B)t^2 + 2Bt\right)e^t$$

$$= \left(At^3 + (6A+B)t^2 + (6A+4B)t + 2B\right)e^t$$

$$Y'' - 2Y' + Y = e^t\Big(\cancel{At^3} + \cancel{(6A+B)}t^2 + (6A + \cancel{4B})t + 2B$$
$$- \cancel{2At^3} - \cancel{2(3A+B)}t^2 - \cancel{4B}t$$
$$+ \cancel{At^3} + \cancel{Bt^2}\Big) + C = te^t + 4$$

$$= e^t\left(6At + 2B\right) + C = te^t + 4$$

$\Rightarrow$ $6A = 1$, $2B = 0$, $C = 4$ $\Rightarrow$ $A = 1/6$, $B = 0$, $C = 4$

So the general solution is

$$y(t) = c_1e^t + c_2te^t + \tfrac{1}{6}t^3e^t + 4$$

$$y'(t) = c_1e^t + c_2e^t + c_2te^t + \tfrac{1}{2}t^2e^t + \tfrac{1}{6}t^3e^t$$

$y(0) = c_1 + 4 = 1$ $\quad c_1 = -3$

$y'(0) = c_1 + c_2 = 1$ $\quad c_2 = 4$

and the particular sol'n is:

$$\boxed{y(t) = -3e^t + 4te^t + \tfrac{1}{6}t^3e^t + 4}$$

6.) Variation of Parameters:

Char. Eqn: $r^2+9=0$

$r=\pm 3i$

$y_1 = \cos(3t)$

$y_2 = \sin(3t)$

$g(t) = 9\sec^2(3t)$

$$W(y_1,y_2) = \begin{vmatrix} \cos(3t) & \sin(3t) \\ 3\sin(3t) & -3\cos(3t) \end{vmatrix} = 3\left(\cos^2(3t)+\sin^2(3t)\right) = 3.$$

$$y(t) = -\cos(3t)\underbrace{\int \frac{9\sin(3t)\sec^2(3t)}{3}\,dt}_{I_1} + \sin(3t)\underbrace{\int \frac{9\cos(3t)\sec^2(3t)}{3}\,dt}_{I_2}$$

$$I_1 = 3\int \frac{\sin(3t)}{\cos^2(3t)}\,dt = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C_1 = -\sec(3t) + C_1$$

$u = \cos(3t)$

$du = 3\sin(3t)\,dt$

$$I_2 = \int \frac{3\cos(3t)}{\cos^2(3t)}\,dt = \int 3\sec(3t)\,dt = \ln|\sec(3t)+\tan(3t)| + C_2$$

So the general solution is:

$$y(t) = -\cos(3t)\left(-\sec(3t)+C_1\right) + \sin(3t)\left(\ln|\sec(3t)+\tan(3t)| + C_2\right)$$

$$\boxed{y(t) = 1 + \sin(3t)\ln|\sec(3t)+\tan(3t)| + C_1\cos(3t) + C_2\sin(3t)}$$

7.) $r^4+2r^2+1 = (r^2+1)^2$

So $r = +i, +i, -i, -i$

$y_1 = \cos t$

$y_2 = t\cos t$

$y_3 = \sin t$

$y_4 = t\sin t$

$Y_1(t) = A$

$Y_2(t) = B\cos(2t) + C\sin(2t)$

$Y_3(t) = De^{-t}$

So our guess should be:

$$\boxed{Y(t) = A + B\cos(2t) + C\sin(2t) + De^{-t}}$$

8.) $r^3 - r = 0$

$r(r^2-1) = 0$

$r = 0,\ r = 1,\ r = -1$

$y_1 = 1$

$y_2 = e^t$

$y_3 = e^{-t}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 1 + 1 = 2$$

$W(y_1, y_2) = e^t$

$W(y_1, y_3) = e^{-t}$

$W(y_2, y_3) = 2$

so the solution is:

$$\boxed{y(t) = 1 \int \frac{2 \csc t}{2}\, dt - e^t \int \frac{e^{-t} \csc t}{2}\, dt + e^{-t} \int \frac{e^t \csc t}{2}\, dt}$$