M555 - Differential Equations

Unit III Review Guide - Solutions

1.a. $f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \quad \xrightarrow{x_0 = 0} \quad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$\Rightarrow \quad f^{(n)}(0) = f^{(2k+1)}(0) = (-1)^k$ for n odd

$f^{(n)}(0) = 0$ for n even

so, $\boxed{\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}$

1.b. $f(x) = x^2 \qquad f(-1) = 1$

$f'(x) = 2x \quad \xrightarrow{x_0 = -1} \quad f'(-1) = -2$

$f''(x) = 2 \qquad f''(-1) = 2$

$f'''(x) = 0$

$\Rightarrow \quad x^2 = 1 - 2(x-1) + \frac{2}{2}(x-1)^2$

or $\boxed{x^2 = 1 - 2(x-1) + (x-1)^2}$

1.c. $\frac{1}{1+x} = \frac{1}{1+(x-2)+2} = \frac{1}{3+(x-2)} = \frac{1/3}{1-\left(-\frac{(x-2)}{3}\right)} = \frac{a}{1-r}$ Geometric Series!

so $\frac{1}{1+x} = \sum_{k=0}^{\infty} \frac{1}{3}\left(-\frac{(x-2)}{3}\right)^k = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(x-2)^k}$

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2. Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|\frac{\cancel{(n+1)}\,\cancel{n!}}{\cancel{(n+1)}(n+1)^n} \cdot \frac{\overset{\text{whoops}}{\cancel{n^n}}}{\cancel{n!}}\right| = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n}$$

$$= \left(1+\frac{1}{n}\right)^{-n} \xrightarrow{n\to\infty} e^{-1} = \frac{1}{e} = L$$

so the radius of convergence is $\boxed{R = \frac{1}{L} = e}$.

3. $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$

$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n$

$y'' - xy' - y =$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Recurrence Relation: $2a_2 - a_0 = 0 \rightarrow \boxed{a_2 = \frac{1}{2} a_0}$

and $(n+2)(n+1) a_{n+2} - n a_n - a_n = 0$

$$\Rightarrow a_{n+2} = \frac{(n+1)}{(n+2)(n+1)} a_n \Rightarrow \boxed{a_{n+2} = \frac{1}{n+2} a_n}$$

for $n \geq 1$.

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4. $y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$(1-x) y'' + xy' - y = (1-x) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} + n a_n - a_n \right] x^n = 0$$

RR: $\boxed{a_2 = \frac{1}{2} a_0}$ and $\boxed{a_{n+2} = \frac{n(n+1) a_{n+1} - (n-1) a_n}{(n+2)(n+1)}}$ for $n \geq 1$

4, cntd. $a_0 = -3$

$a_1 = 2$

$a_2 = \frac{1}{2} a_0 = \frac{-3}{2}$

$$a_3 = \frac{2a_2 - 0\,a_1}{3 \cdot 2} = \frac{1}{3} a_2 = -\frac{1}{2}$$

so the first 4 terms of the solution are:

$$\boxed{y = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3}$$

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5. $(x^2-2x-3)y'' + xy' + 4y = 0$

$$\Rightarrow \quad y'' + \frac{x}{x^2-2x-3} y' + \frac{4}{x^2-2x-3} y = 0$$

$x^2 - 2x - 3 \neq 0$

$(x-3)(x+1) \neq 0$

$x \neq 3, \quad x \neq -1.$

Therefore, the minimum radius of conv. about x_0^i is:

$R^1 = 1$ about $x_0^1 = 4$, $R^2 = 3$ about $x_0^1 = 4$, and $R^3 = 0$ about $x_0^3 = 0$.

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6. $\varphi(1) = 2, \quad \varphi'(1) = 0.$

$$\varphi''(x) = \frac{-(1+x)y' - 3\ln x\, y}{x^2} \Rightarrow \varphi''(1) = \frac{-2\varphi'(1) - 3\cdot 0 \cdot \varphi(1)}{1^2} = -2\varphi'(1) = 0.$$

$$\varphi'''(x) = (\varphi''(x))' = \frac{x^2\left(-y' - (1+x)y'' - \frac{3}{x} y - 3\ln x\, y'\right) - \left(-(1+x)y' - 3\ln x\, y\right)2x}{x^4}$$

$$\varphi'''(1) = \frac{-\varphi'(1) - 2\varphi''(1) - 3\varphi(1) - 0 - \left(-2\varphi'(1) - 0\right)\cdot 2}{1} = -3\varphi(1) = -6.$$

Repeat this for $\varphi^{(4)}$ and $\varphi^{(5)}$. The equation on the exam won't be so gross, and won't ask for $\varphi^{(4)}$ and $\varphi^{(5)}$.

7. Singular points are values of x s.t. $x \sin x = 0$.

so, $x = 0$ and $x = n\pi$ for all integers $n = \pm 1, \pm 2, \pm 3, \ldots$

To check regularity, take

$$\lim_{x \to \infty} x \frac{3}{x \sin x} \qquad \text{then also} \qquad \lim_{x \to \infty} (x - n\pi) \frac{3}{x \sin x}$$

and

$$\lim_{x \to \infty} x^2 \frac{x}{x \sin x} \qquad \qquad \lim_{x \to \infty} (x - n\pi)^2 \frac{x}{x \sin x}$$

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8. a.)

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt = -\frac{e^{-st}}{s} n t^{n-1} \Big|_{t=0}^{\infty} + n \int_0^\infty \frac{e^{-st}}{s} t^{n-1} \, dt$$

$$= \text{repeat IbP until } t\text{'s are gone} \ldots$$

$$= \frac{-e^{-st}}{s} n t^{n-1} - \frac{e^{-st}}{s^2} n(n-1) t^{n-2} - \cdots + \frac{n!}{s^n} \int_0^\infty e^{-st} dt$$

$$= \boxed{\frac{n!}{s^{n+1}} \quad \text{for } s > 0.}$$

b.) $\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \sin t \, dt = F$. Apply IbP twice

$u = e^{-st}$ $\quad dv = \sin t \, dt$
$du = -s e^{-st} dt$ $\quad v = -\cos t$

$u = e^{-st}$ $\quad dv = \cos t \, dt$
$du = -s e^{-st} dt$ $\quad v = \sin t$

$$F = -\cos t \, e^{-st} - s \int e^{-st} \cos t \, dt$$

$$= -\cos t \, e^{-st} - s \left(e^{-st} \sin t + s \int_0^\infty e^{-st} \sin t \, dt \right)$$

$$F = -e^{-st} (\cos t + s \sin t) - s^2 F$$

$$(1 - s^2) F = -e^{-st} (\cos t + s \sin t) \Big|_0^\infty \Rightarrow F = \frac{e^{-st}(\cos t + s \sin t)}{s^2 - 1} \Bigg|_{t=0}^{\infty} = \boxed{\frac{1}{s^2 - 1}}$$

for $s > 0$

8.c.) $\mathcal{L}\{\sinh t\} = \int_0^\infty e^{-st}\sinh t\,dt = \frac{1}{2}\int_0^\infty e^{-st}e^{t} - e^{-st}e^{-t}\,dt$

$$= \frac{1}{2}\int_0^\infty e^{-(s-1)t}\,dt - \frac{1}{2}\int_0^\infty e^{-(s+1)t}\,dt$$

$$= \frac{1}{2}\cdot\frac{-1}{s-1}\cdot e^{-(s-1)t}\Big|_0^\infty - \frac{1}{2}\frac{-1}{s+1}e^{-(s+1)t}\Big|_0^\infty$$

$$= -\frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} \quad \text{for } s > +1.$$

$$= \frac{1}{2}\left(\frac{-(s+1)+(s-1)}{s^2-1}\right) = \frac{1}{2}\left(\frac{-2}{s^2-1}\right) = \frac{1}{1-s^2}.$$

9.a.) $F(s) = \frac{2s+1}{s^2-2s+2} = \frac{2s+1}{(s-1)^2+1} = \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$

so $f(t) = 2e^{t}\cos t + 3e^{t}\sin t$

9.b.) $F(s) = \frac{2s-3}{s^2+4s-4} = \frac{2s-3}{(s+2)^2-8} = \frac{2(s+2)}{(s+2)^2-8} + \frac{-7}{(s+2)^2-8}$

so $f(t) = 2e^{-2t}\cosh(\sqrt{8}\,t) - \left(\frac{7}{\sqrt{8}}\right)e^{-2t}\sinh(\sqrt{8}\,t)$.

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10. $\mathcal{L}\{y'' - y' - 6y\} = 0 \Rightarrow s^2Y - sy(0) - y'(0) - sY - y(0) - 6Y = 0$

$(s^2-s-6)Y = s - 1 + 1$

so $Y = \frac{s}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2}$

PFD: $s = (s+2)A + (s-3)B$

$3 = 5A \Rightarrow A = 3/5$

$-2 = -5B \Rightarrow B = 2/5$

$$\boxed{y = \frac{3}{5}e^{3t} + \frac{2}{5}e^{-2t}}$$

11. $\mathcal{L}\{y^{(4)} - 4y\} = 0 \rightarrow s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4Y = 0$

$$(s^2-4)Y = 1s^3+0s^2-2s+0$$

$$Y = \frac{s^3-2s}{s^2-4} = \frac{A}{s-\sqrt{2}} + \frac{B}{s+\sqrt{2}} + \frac{Cs+D}{s^2+2}$$

$$s^3-2s = A(s+\sqrt{2})(s^2+2) + B(s-\sqrt{2})(s^2+2) + (Cs+D)(s^2-2)$$

$s=\sqrt{2}$: $2\sqrt{2}-2\sqrt{2} = 2\sqrt{2}(4)A \rightarrow A = 0.$

$s=-\sqrt{2}$: $-2\sqrt{2}+2\sqrt{2} = B(-2\sqrt{2})(4) \rightarrow B=0$

$$s^3-2s = Cs^3+Ds^2-2Cs-2D$$

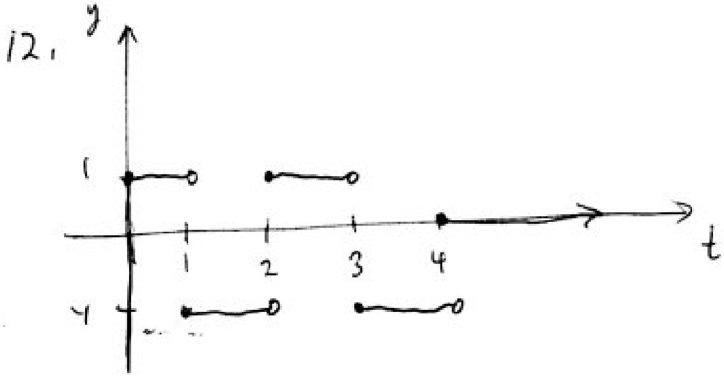
$$\begin{cases} C=1 \\ D=0 \end{cases}$$

so $Y = \frac{s}{s^2+2}$ and $\boxed{y = \cos(\sqrt{2}t)}$

Could have done this much easier:

$$\frac{s^3-2s}{s^2-4} = \frac{s(\cancel{s^2-2})}{(s^2+2)(\cancel{s^2-2})}$$

12.

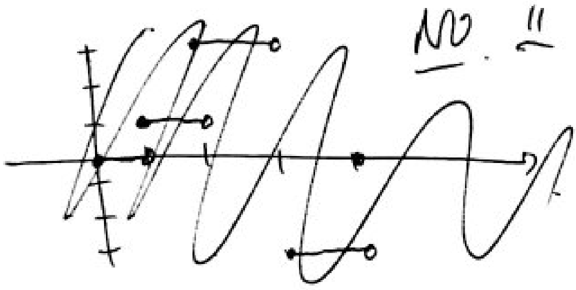


$$\boxed{f(t) = u_0(t) - 2u_1(t) + 2u_2(t) - 2u_3(t) + u_4(t)}$$

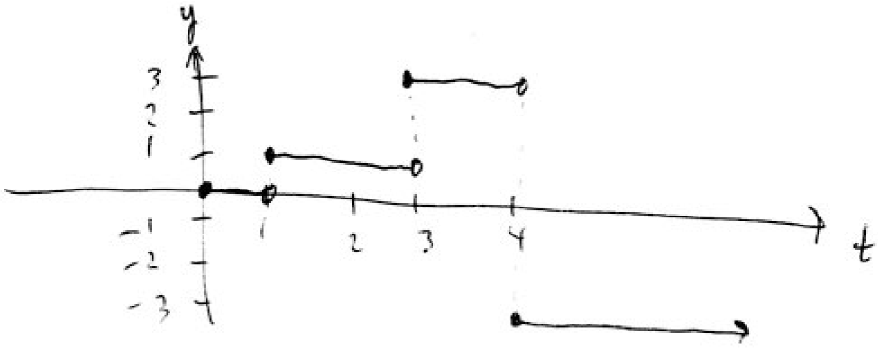
$$\boxed{F(s) = \frac{1-2e^{-s}+2e^{-2s}-2e^{-3s}+e^{-4s}}{s}}$$

13. $\boxed{F(s) = \mathcal{L}\{f(t)\} = \dfrac{e^{-s} + 2e^{-3s} - 6e^{-4s}}{s}}$

NO !!



$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 3 \\ 3 & 3 \le t < 4 \\ -3 & t \ge 4 \end{cases}$$



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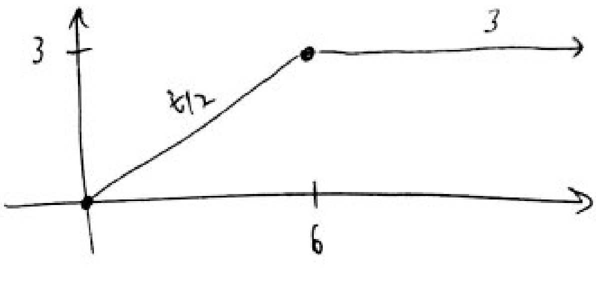
14. $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s} - \dfrac{e^{-3s}}{s} - \dfrac{e^{-4s}}{s}\right\}$

$= u_1(t) + u_2(t) - u_3(t) - u_4(t)$

$$= \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 2 & 2 \le t < 3 \\ 1 & 3 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

15.



$f(t) = u_0(t)\,\frac{t}{2} - u_3(t)\,\frac{(t-3)}{2}$

$F(s) = \frac{1}{2} \cdot \frac{1}{s^2} - \frac{e^{-3s}}{2}\,\frac{1}{s^2} = \frac{1 - e^{-3s}}{2s^2}$

so the DE becomes $s^2 Y - s y(0) - y'(0) + Y = \dfrac{1 - e^{-3s}}{2s^2}$

$(s^2+1)\,Y = \dfrac{1 - e^{-3s}}{2s^2} + 1 = \dfrac{1 + 2s^2 - e^{-3s}}{2s^2}$

or $Y = \dfrac{1 + 2s^2 - e^{-3s}}{2s^2(s^2+1)}$

$$Y = \underbrace{\frac{1+2s^2}{2s^2(s^2+1)}}_{Y_1} - \underbrace{\frac{e^{-3s}}{2s^2(s^2+1)}}_{Y_2}$$

Y_1: $\dfrac{1+2s^2}{2s^2(s^2+1)} = \dfrac{1}{2}\left(\dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1}\right)$

$$1 + 2s^2 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$2s^2 + 1 = (A+C)s^3 + (B+D)s^2 + (As) + B$$

$A = 0 \Rightarrow C = 0$

$B = 1 \Rightarrow D = 2 - 1 = 1.$

So, $Y_1 = \dfrac{1}{2}\left(\dfrac{1}{s^2} + \dfrac{1}{s^2+1}\right) = \dfrac{1}{s^2} + \dfrac{1}{s^2+1}$

and $\boxed{y_1 = \frac{1}{2}t + \frac{1}{2}\sin t}$

Y_2: $\dfrac{1}{2s^2(s^2+1)} = \dfrac{1}{2}\left(\dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1}\right)$

$$1 = (A+C)s^3 + (B+D)s^2 + As + B$$

$B = 1 \Rightarrow D = -1$

$A = C = 0$

So, $Y_2 = -\dfrac{e^{-3s}}{2}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+1}\right)$

$= -\dfrac{1}{2}\dfrac{e^{-3s}}{s^2} + \dfrac{1}{2}\dfrac{e^{-3s}}{s^2+1}$

and $y_2 = \left(-\frac{1}{2}u_3(t)(t-3) + \frac{1}{2}u_3(t)\sin(t-3)\right)$

The total sol'n is

$$\boxed{y(t) = \frac{1}{2}t - \frac{1}{2}u_3(t)(t-3) + \frac{1}{2}\sin t + \frac{1}{2}u_3(t)\sin(t-3)}$$