Name: Key
M555: Differential Equations I (Spring 2018)
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Unit I Exam: Chapters 1 and 2

WICHITA STATE UNIVERSITY

Read and follow all instructions, and show enough work. You may not use any notes or electronic devices. All you need is a pencil and your brain!

1. Solve the initial value problem.

$$\begin{cases} y' + \dfrac{1}{t} y = e^{2t}, \\ y\left(\tfrac{1}{2}\right) = 4. \end{cases}$$

Linear!

$p(t) = 1/t$

$\int p\,dt = \ln t$

$\mu(t) = e^{\ln t} = t$

since $y(\tfrac{1}{2}) = 2(C) = 4$.

$$y(t) = \frac{1}{t}\left(\int_{\frac{1}{2}}^{t} u e^{2u}\,du + 2\right)$$

$$= \frac{1}{t}\left(\frac{1}{2}ue^{2u} - \frac{1}{4}e^{2u}\Big|_{1/2}^{t} + 2\right)$$

$$= \frac{1}{t}\left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} - \cancel{\frac{1}{4}e} + \cancel{\frac{1}{4}e} + 2\right)$$

$$\boxed{y(t) = \frac{1}{2}e^{2t} - \frac{e^{2t}}{4t} + \frac{2}{t}}$$

2. Find the particular solution of the initial value problem.

$$\begin{cases} (2xy - y\cos(xy) + \ln y)\,dx + \left(x^2 - x\cos(xy) + \dfrac{x}{y}\right)dy = 0, \\ y(0) = 1 \end{cases}$$

$M = 2xy - y\cos(xy) + \ln y \Rightarrow \dfrac{\partial M}{\partial y} = 2x + xy\sin(xy) + \dfrac{1}{y}$

$N = x^2 - x\cos(xy) + \dfrac{x}{y} \Rightarrow \dfrac{\partial N}{\partial x} = 2x + xy\sin(xy) + \dfrac{1}{y}$

Exact!

$\varphi = \int M\,dx = \int 2xy - y\cos(xy) + \ln y\,dx = x^2y - \sin(xy) + x\ln y + C_1(y)$

$\varphi = \int N\,dy = \int x^2 - x\cos(xy) + \frac{x}{y}\,dy = x^2y - \sin(xy) + x\ln y + \underbrace{C_2(x)}_{=0}$

General sol'n:

$$\varphi(x,y) = x^2y - \sin(xy) + x\ln y = C$$

Plug in initial condition:

$$\varphi(0,1) = 0 - \sin(0) + 0\cdot\ln 1 = 0.$$

The particular sol'n is

$$\boxed{x^2y - \sin(xy) + x\ln y = 0}$$

3. Find the general solution of the differential equation.

$$(2y + x^2 y)y' = x$$

Separable!

$$y(x^2+2)\frac{dy}{dx} = x$$

$$\Rightarrow \int y\, dy = \int \frac{x}{x^2+2}\, dx \qquad \begin{array}{l} u = x^2+2 \\ du = 2x\,dx \end{array}$$

$$\frac{1}{2}y^2 = \frac{1}{2}\ln(x^2+2) + C$$

$$\Rightarrow \boxed{y(x) = \pm\sqrt{\ln(x^2+2)+C}}$$

4. Consider the differential equation,

$$(2y+1)+\left(x-\frac{y}{x}\right)y'=0.$$

a.) Show that the DE is not exact as written.

$M_1 = 2y+1 \qquad \frac{\partial M_1}{\partial y} = 2$

$N_1 = x - y/x \qquad \frac{\partial N_1}{\partial x} = 1 + y/x^2$

$\Big\} \neq$

b.) Show that the DE is exact when multiplied by the integrating factor $\mu(x, y) = x$, and use this fact to find the general solution of the DE.

$M = xM_1 = 2xy + x \qquad \partial M/\partial y = 2x$

$N = xN_1 = x^2 - y \qquad \partial N/\partial x = 2x$

$\Big\}$ Exact!

$$\varphi = \int M\,dx = \int 2xy + x\,dx = x^2y + \tfrac{1}{2}x^2 + C_1(y)$$

$$\varphi = \int N\,dy = \int x^2 - y\,dy = x^2y - \tfrac{1}{2}y^2 + C_2(x)$$

so the solution is:

$$\boxed{x^2y + \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 = C}$$

or

$$(y^2 - 2x^2y + x^4) = C + x^2 + x^4$$

$$(y - x^2)^2 = C + x^2 + x^4$$

so, $\boxed{y(x) = x^2 \pm \sqrt{x^4 + x^2 + C}}$

5. Consider the initial value problem (IVP)

$$\begin{cases} y' = 2xy^2, \\ y(0) = -1. \end{cases}$$

a.) Use the Fundamental Existence and Uniqueness Theorem to verify that a solution exists for the given initial data.

$$\left.\begin{aligned} f(x,y) &= 2xy^2 \\ \frac{\partial f}{\partial y} &= 4xy \end{aligned}\right\} \text{ both continuous for all } (x,y).$$

b.) Solve the IVP and clearly state the domain of the solution.

Separable!

$$\int \frac{1}{y^2}\,dy = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + C \xrightarrow{y(0)=-1} 1 = 0^2 + C \Rightarrow C = 1$$

$$\boxed{y = \frac{-1}{x^2+1}}$$

the domain of the solution is all real numbers.

6. Consider the initial value problem

$$\begin{cases} y' - y = 1 - t, \\ y(0) = 0. \end{cases}$$

a.) Use Picard's Method of Successive Approximations with $\varphi_0 = 0$ to find φ_1, φ_2, φ_3, and a general formula for φ_n.

$f(t, y) = y + 1 - t$

$\varphi_0 = 0$

$\varphi_1 = \int_0^t 0 + 1 - u \, du = u - \frac{1}{2}u^2 \Big|_0^t = t - \frac{1}{2}t^2$

$\varphi_2 = \int_0^t \cancel{u} - \frac{1}{2}u^2 + 1 - \cancel{u} = u - \frac{1}{3!}u^3 \Big|_0^t = t - \frac{1}{3!}t^3$

$\varphi_3 = \int_0^t \cancel{u} - \frac{1}{3!}u^3 + 1 - \cancel{u} \, du = u - \frac{1}{4!}u^4 \Big|_0^t = t - \frac{1}{4!}t^4$

the general n^{th} approximation is

$\varphi_n(t) = t - \frac{1}{(n+1)!}t^{n+1}$

b.) Find the solution, $\varphi = \lim_{n \to \infty} \varphi_n$.

Therefore the solution is

$\varphi(t) = \lim_{n \to \infty} \left(t - \frac{1}{(n+1)!}t^{n+1} \right) = t - \lim_{n \to \infty} \frac{1}{(n+1)!} t^{n+1} = t - 0 = t.$

So $\boxed{\varphi(t) = t}$ is the solution of the IVP