Name: Key

M555: Differential Equations I (Spring 2018)
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Good Problems 3 – Sections 2.4, 2.8, and 2.6

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Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

1. Consider the differential equation

$$y' = \frac{t-y}{2t+5y}.$$

Determine where in the ty-plane the hypotheses of the Fundamental Existence and Uniqueness Theorem are satisfied.

$f(t,y) = \dfrac{t-y}{2t+5y}$ is not continuous when $2t+5y=0$, or $y = -\frac{2}{5}t$.

$\dfrac{\partial f}{\partial y} = \dfrac{(2t+5y)(-1)-(t-y)(5)}{(2t+5y)^2} = \dfrac{-7t}{(2t+5y)^2}$ has the same restriction.

So the hypotheses of the FEUT are satisfied so long as $y \neq -2/5\, t$.

2. Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = t^2 + y^2, \\ y(1) = 2. \end{cases}$$

Make a change of variables to transform this IVP into an equivalent problem with the initial data $y(0) = 0$.

$y - 2 \mapsto Y \Rightarrow y = Y + 2$

$t - 1 \mapsto T \Rightarrow t = T + 1$

The DE becomes:

$$\begin{cases} \dfrac{dY}{dT} = (T+1)^2 + (Y+2)^2 \\ Y(0) = 0 \end{cases}$$

3. Consider the initial value problem

$$\begin{cases} y' = t^2 y - t, \\ y(0) = 0. \end{cases}$$

Use Picard's method of successive approximations with $\varphi_0(t) = 0$ to:

a.) Determine formulas for φ_1, φ_2, φ_3, and φ_4.

b.) Use a computer to plot $\varphi_0, \ldots, \varphi_4$ on the same set of axes. (Include a printed graph with your homework submission.)

c.) Show that the sequence $\{\varphi_n\}$ converges.

a.) $\varphi_0 = 0.$

$\varphi_1 = \int_0^t 0 - u\,du = -\frac{1}{2}t^2$

$\varphi_2 = \int_0^t u^2(-\frac{1}{2}u^2) - u\,du = -\frac{1}{2\cdot 5}t^5 - \frac{1}{2}t^2$

$\varphi_3 = \int_0^t u^2(-\frac{1}{2\cdot 5}u^5 - \frac{1}{2}u^2) - u\,du = \frac{-1}{2\cdot 5\cdot 8}t^8 - \frac{1}{2\cdot 5}t^5 - \frac{1}{2}t^2$

$\varphi_4 = \int_0^t u^2\left(\frac{-1}{2\cdot 5\cdot 8}u^8 - \frac{1}{2\cdot 5}u^5 - \frac{1}{2}u^2\right) - u\,du = \frac{-1}{2\cdot 5\cdot 8\cdot 11}t^{11} - \frac{1}{2\cdot 5\cdot 8}t^8 - \frac{1}{2\cdot 5}t^5 - \frac{1}{2}t^2$

b.) Use Geogebra or other free app.

c.) $\varphi_n = \sum_{k=1}^{n} \frac{-1}{(3k-1)\cdot(3k-4)\cdots} t^{3k-1}$

Apply the Ratio Test:

$$\left|\frac{\hat{\varphi}_{n+1}}{\varphi_n}\right| = \left|\frac{+\cancel{(3n-1)}\cdot\cancel{(3n-4)}\cdot\cancel{(3n-7)}\cdots}{+(3(n+1)-1)\cdot\cancel{(3n-1)}\cdot\cancel{(3n-4)}\cdots} \cdot \frac{t^{(3(n+1)-1)}}{t^{3n-1}}\right| = \left|\frac{t^3}{3n+2}\right|$$

$$\lim_{n\to\infty}\left|\frac{\hat{\varphi}_{n+1}}{\varphi_n}\right| = \lim_{n\to\infty}\left|\frac{t^3}{3n+2}\right| = 0<1$$ so the sequence converges.

4. Consider the initial value problem,

$$\begin{cases} y' - y = 1 - t, \\ y(0) = 0. \end{cases} \qquad f(t,y) = y + 1 - t$$

a.) Use Picard's method of successive iterations to find a formula for the n^{th} approximation, $\varphi_n(t)$.

b.) Compute $\lim_{n\to\infty} \varphi_n(t)$ to find a formula for the actual solution $\varphi(t)$.

c.) Solve the IVP using the method of integrating factors and verify that your solutions using each method agree.

a.) $\varphi_0 = 0$

$\varphi_1 = \int_0^t 0 + 1 - u\,du = u - \frac{1}{2}u^2 \Big|_0^t = t - \frac{1}{2}t^2$

$\varphi_2 = \int_0^t \cancel{u} - \frac{1}{2}u^2 + 1 - \cancel{u}\,du = t - \frac{1}{3!}t^3$

$\varphi_3 = \int_0^t \cancel{u} - \frac{1}{3!}u^3 + 1 - \cancel{u}\,du = t - \frac{1}{4!}t^4$

$\vdots$

$\boxed{\varphi_n = t - \frac{1}{(n+1)!}t^{n+1}}$

b.) $\varphi(t) = \lim_{n\to\infty} \varphi_n = \lim_{n\to\infty}\left(t - \frac{1}{(n+1)!}t^{n+1}\right) = t - 0 = t.$

$\boxed{\varphi(t) = t.}$

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c.) Linear!

$p(t) = -1$

$\mu(t) = e^{-t}$ ← be careful w/ the signs!

$y(t) = e^{t}\left(\int_0^t (1-u)\,e^{-u}\,du\right) = e^{t}\left(-e^{-t} + (te^{-t} + e^{-t}) + (1)\right)$

$= e^{t}\left(te^{-t}\right) = t.$ ✓

5. Verify that the differential equation is not exact as written, but is exact when multiplied by the given integrating factor, $\mu(x, y)$. Then use the integrating factor to solve the DE.

$$\begin{cases} \left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right) + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)\dfrac{dy}{dx} = 0, \\ \mu(x,y) = ye^{x} \end{cases}$$

$$\left.\begin{aligned} &M_1 = \frac{\sin y}{y} - 2e^{-x}\sin x \qquad \frac{\partial M_1}{\partial y} = +\frac{y\cos y - \sin y}{y^2} \\ &N_1 = \frac{\cos y}{y} + \frac{1}{y}\left(2e^{-x}\cos x\right) \qquad \frac{\partial N_1}{\partial x} = \frac{1}{y}\left(-2e^{-x}\cos x - 2e^{-x}\sin x\right) \end{aligned}\right\} \neq$$

$$\left.\begin{aligned} &M = \mu M_1 = e^{x}\sin y - 2y\sin x \qquad \partial M/\partial y = e^{x}\cos y - 2\sin x \\ &N = \mu N_1 = e^{x}\cos y + 2\cos x \qquad \partial N/\partial x = e^{x}\cos y - 2\sin x \end{aligned}\right\} \overset{!}{=} \text{ exact}$$

$$\varphi = \int M\,dx = \int e^{x}\sin y - 2y\sin x\,dx = e^{x}\sin y + 2y\cos x + C_1(y)$$

$$\varphi = \int N\,dx = \int e^{x}\cos y + 2\cos x\,dy = e^{x}\sin y + 2y\cos x \underbrace{+C_2(x)}_{=0}$$

So, the general sol'n is

$$\varphi(x,y) = \boxed{e^{x}\sin y + 2y\cos x = C}$$