Name: Key

M555: Differential Equations I (Spring 2018)
Instructor: Justin Ryan
Good Problems 4: Sections 3.1 and 3.2

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Solve the initial value problem

$$\begin{cases} y'' + 5y' + 3y = 0, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

characteristic eqn: $r^2 + 5r + 3 = 0$

$\left(r^2 + 5r + \frac{25}{4}\right) = \frac{25}{4} - \frac{3\cdot 4}{4}$

$\left(r + \frac{5}{2}\right)^2 = \frac{13}{4}$

$r = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$

$r_+ = -\frac{5}{2} + \frac{\sqrt{13}}{2}$

$r_- = -\frac{5}{2} - \frac{\sqrt{13}}{2}$

$y = C_1 e^{r_+ t} + C_2 e^{r_- t}$

$y' = r_+ C_1 e^{r_+ t} + r_- C_2 e^{r_- t}$

$y(0) = C_1 + C_2 = 1$

$y'(0) = r_+ C_1 + r_- C_2 = 0$

$C_2 = \frac{-r_+}{r_-} C_1 = \left(\frac{\sqrt{13}-5}{\sqrt{13}+5}\right) C_1$

So,

$\left(1 + \frac{\sqrt{13}-5}{\sqrt{13}+5}\right) C_1 = 1$

$\left(\frac{\sqrt{13}+5+\sqrt{13}-5}{\sqrt{13}+5}\right)^{-1} = C_1$

$C_1 = \frac{\sqrt{13}+5}{2\sqrt{13}}$

and

$C_2 = \frac{\sqrt{13}-5}{2\sqrt{13}}$.

Final sol'n is in the box

$$\boxed{y(t) = \left(\frac{\sqrt{13}+5}{2\sqrt{13}}\right) e^{\left(\frac{-5+\sqrt{13}}{2}\right)t} + \left(\frac{\sqrt{13}-5}{2\sqrt{13}}\right) e^{\left(\frac{-5-\sqrt{13}}{2}\right)t}}$$

2. Solve the initial value problem

$$\begin{cases} 2y'' - 3y' + y = 0, \\ y(0) = 0, \\ y'(0) = 2. \end{cases}$$

characteristic eqn: $2r^2 - 3r + 1 = 0$

$2r^2 - 2r - r + 1 = 0$

$2r(r-1) - (r-1) = 0$

$(2r-1)(r-1) = 0$

$r_1 = 1/2 \quad r_2 = 1$

So, $y = C_1 e^{1/2\, t} + C_2 e^t$

and $y' = \frac{1}{2} C_1 e^{1/2\, t} + C_2 e^t$

$y(0) = C_1 + C_2 = 0 \rightarrow C_2 = -C_1$

$y'(0) = \frac{1}{2} C_1 + C_2 = 2 \rightarrow C_2 = 2 - \frac{1}{2} C_1$

$-C_1 = 2 - 1/2\, C_1 \rightarrow -\frac{1}{2} C_1 = 2 \rightarrow C_1 = -4$.

So $C_2 = 4$, and

$$\boxed{y(t) = -4 e^{1/2\, t} + 4 e^t}$$

3. Find a second order linear homogeneous differential equation with constant coefficients for which the general solution is

$$y(t) = C_1 e^{-t/2} + C_2 e^{-2t}.$$

$r_1 = -1/2$, $r_2 = -2$

characteristic eqn is $(r - r_1)(r - r_2) = 0$

$(r + 1/2)(r + 2) = 0$

$r^2 + \frac{5}{2} r + 1 = 0$

or $2r^2 + 5r + 1 = 0$

so the DE is

$$\boxed{2y'' + 5y' + y = 0}$$

4. Show that the functions $y_1 = e^{rt}$ and $y_2 = te^{rt}$ are linearly independent.

$$W(e^{rt}, te^{rt}) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{vmatrix} = e^{2rt} + rte^{2rt} - rte^{2rt} = e^{2rt} \neq 0.$$

Since the Wronskian is not zero, therefore the functions are linearly independent.